How to put a stop to domination once and for all (and what comes afterwards)

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Abstract

We study the generation of C^1-robustly non-dominated dynamics and some relevant features associated to this sort of dynamics. For that we study heterodimensional cycles and introduce a bifurcation called heterodimensional tangency.

Keywords: dominated splitting, heterodimensional cycle, heterodimensional tangency, homoclinic class, homoclinic tangency, index of a saddle, minimal set, Newhouse's coexistence phenomenon.
MSC 2000: 37C05, 37C20, 37C25, 37C29, 37D30.

1 Introduction

This paper is devoted to the analysis of the dynamical consequences of C^1-non-dominated dynamics and to explain the generation of such non-dominated dynamics.

We begin by discussing briefly Smale's density conjecture for hyperbolic dynamics (*the C^r-diffeomorphisms satisfying the Axiom A and the no-cycles condition form a dense subset of* $\mathrm{Diff}^r(M)$, *where M is a closed manifold*)[1]. Nowadays, it is known that this conjecture is false for manifolds of dimension bigger than or equal to three in any topology and for C^r surface diffeomorphisms when $r \geq 2$. The conjecture is true in dimension one (any C^r-topology, $r \geq 1$). Finally, it remains open for C^1-diffeomorphisms defined on surfaces.

In very rough terms, the Axiom A condition means that the periodic points are dense in the whole non-wandering set, that they are all hyperbolic, and, moreover that the hyperbolic structures on these periodic points (stable and unstable directions) fit together nicely. Due to dimensional reasons, the proof of Smale's conjecture is rather simple for circle diffeomorphisms: the non-wandering set (typically) consists of finitely many hyperbolic (attracting or repelling) periodic points and there are no homoclinic phenomena. Thus it is not necessary to glue hyperbolic structures of periodic points. In higher dimensions, three different sort of obstructions appear for gluing these hyperbolic

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[1] In this introduction, we omit technical and precise definitions, we refer to the corresponding section for details.

structures: (i) *lack of domination* or/and arbitrarily *small angles* between the invariants stable and unstable directions of periodic points, (ii) *dimension variability,* that is, coexistence in the same elementary piece of dynamics (homoclinic class) of saddles having different *indices* (dimension of the unstable direction), and (iii) existence of infinitely many elementary pieces of dynamics, the so-called *Newhouse's coexistence phenomenon.* We will see that in many cases the lack of domination and the coexistence phenomenon occur simultaneously and that there is a strong relation between them. Let us also observe that the dimension variability can only occur in manifolds of dimension greater than or equal to three.

Let us first briefly outline the proof of Smale's conjecture for circle diffeomorphisms. For simplicity, let us consider diffeomorphisms f preserving the orientation, $f \in \mathrm{Diff}^{+,r}(\mathbb{S}^1)$, $r \geq 1$. First, one has that the diffeomorphisms f having rational rotation number $\rho(f)$ are dense in $\mathrm{Diff}^{+,r}(\mathbb{S}^1)$. Recall that a diffeomorphism has rational rotation number if, and only if, it has some periodic orbit. Note also that a periodic point can be made hyperbolic after an arbitrarily small perturbation. These two facts imply that the diffeomorphisms having rational number contains an open and dense subset $\mathcal{H}$ of $\mathrm{Diff}^{+,r}(\mathbb{S}^1)$. Moreover, the properties of the rotation number (in fact, the existence of hyperbolic periodic points) imply that the set $\mathcal{H}$ can be chosen such that the rotation number is locally constant in $\mathcal{H}$. The next step is to see that the primary periods of the periodic points of diffeomorphisms $f \in \mathcal{H}$ are bounded (in fact, the same), thus every periodic point of f can be made hyperbolic after an arbitrarily small perturbation. This, in particular, implies that there are only finitely many periodic points.

The final step is to see that, in this case, the non-wandering set of f is just the union these finitely many periodic orbits. Indeed, given any attracting orbit of f note that its stable manifold consists of finitely many open intervals whose boundary consists of periodic orbits (which are necessarily repelling by the hyperbolicity hypothesis). It is now easy to see that the complement of the union of the stable manifolds of the finite set of attracting orbits of f consists of (finitely many) repelling orbits. This immediately implies that the non-wandering set of f is a finite set of hyperbolic periodic points. Indeed, this implies the density of Morse-Smale systems for circle diffeomorphisms. Note that in dimension one the periodic points are either attracting or repelling, thus there are no cycles.

We observe that the of proof the density of Axiom A maps of the interval is much more intricate (this is due to the existence of homoclinic phenomena and critical points). Firstly, Kozlovski proved in [26] that Axiom A maps are dense in the space of C^k unimodal interval maps, for all $k \in \mathbb{N}$. Secondly, Shen proved in [42] that Axiom A maps are dense in the space of C^2 maps of the interval endowed with the C^2-topology. Finally, the general case (Axiom A maps are dense in the space of C^r maps of the interval, any $r \geq 1$) was settled by Kozlovski-Shen-van Strien in [27]. Previously, [24, 28] proved the density of Axiom A maps in the family of real quadratic polynomials. The proof of these results involve sophisticated tools as, for example, quasi-conformal deformations and the Milnor-Thurston kneading theory. These results are a outstanding contribution toward a positive answer to the question of the density of hyperbolicity for interval maps. Indeed, a better understanding of geometric aspects of the dynamics and the development of a theory of quasi-conformal deformations brought a significant progress to this subject in recent years, which culminated with the works mentioned above. It is worth mentioning that a conjecture of complex density of hyperbolicity was stated by Fatou around 1920 with his belief that *most* rational complex maps are expanding on their Julia set.

In Section 2, we review two different kind of counter-examples to the conjecture of Smale and outline some recent progress in the case of C^1 surface diffeomorphisms. We begin in Section 2.1 by presenting the counter-examples related to the dimension variability (chronologically the first ones and due in the C^1-case to Abraham-Smale, [3]): there are two transitive hyperbolic sets Λ_f and Σ_f having different indices (dimension of the unstable bundle, note that the transitivity of the sets implies that the dimension of the unstable bundle is constant in each hyperbolic set) which are related in a robust way by a *(heterodimensional) cycle*: for every diffeomorphism g close to f, the hyperbolic continuations Λ_g and Σ_g of Λ_f and Σ_f are such that the stable manifold of Λ_g meets the unstable one of Σ_g and vice-versa. This sort of robust cycles are displayed in any C^r-topology, $r \geq 1$, in any manifold of dimension at least three, see [14]. We will see in Section 3 how *heterodimensional cycles* associated to a pair of hyperbolic saddles (i.e., P and Q are saddles having different indices with $W^u(P) \cap W^s(Q) \neq \emptyset$ and $W^s(P) \cap W^u(Q) \neq \emptyset$) generate robust cycles as the ones above. Indeed, heterodimensional cycles will be a key ingredient in most constructions of this paper.

In Section 2.2, we study the counter-examples associated to C^2-robust tangencies (a special case of lack of domination). These examples exhibit C^2-robustly non-dominated dynamics, the main topic of this note. We discuss Newhouse's construction of (non-trivial) hyperbolic sets Λ_f whose stable and unstable manifolds exhibit tangencies in a robust way (Theorem 1): for every diffeomorphisms g which is C^2-close to f, there are points $x, y \in \Lambda_g$ such that the stable leaf of x is tangent to the unstable leaf of y. This construction involves distortion estimates and it is typically C^2. We see how this construction is closely related to homoclinic bifurcations and generates infinitely many sinks and/or sources in a persistent way (Newhouse's coexistence phenomenon). We will discuss strong forms of the coexistence phenomenon in the C^1-setting (see Sections 4 and 5) and study its relation to the lack of domination.

Finally, in Section 2.3, we discuss the current state of Smale's conjecture for C^1-surface diffeomorphisms. Let us observe that for proving the conjecture it is not enough to prove the hyperbolicity of each homoclinic classes (of course, this is a necessary condition). *A priori,* a diffeomorphism may have infinitely homoclinic classes being hyperbolic with some persistence (a special case of the coexistence phenomenon). We present a trichotomy result for Smale's conjecture involving the conditions (i) Axiom A plus no-cycles, (ii) persistent homoclinic tangencies, and (iii) infinitely many hyperbolic homoclinic classes, see Theorem 2. Finally, for operating reasons and bearing in mind some recent results, it is interesting to split Smale's conjecture for C^1 surface diffeomorphisms into three sub-conjectures (see Conjecture 1). Positive answers to these *small* conjectures would imply the initial conjecture.

In Section 3, we study the dynamics at heterodimensional cycles. We see how these cycles generate C^1-robustly non-hyperbolic homoclinic classes (i.e., a diffeomorphism f with a saddle P_f such that, for any g close to f, the homoclinic class of the saddle P_g is not hyperbolic, indeed it contains saddles with different indices) and robust cycles as the ones above, see Theorems 3 and 4.

Using heterodimensional cycles, we get a simple form of the coexistence phenomenon in the C^1-topology for three manifolds (analogous to the one obtained by Newhouse for C^2 surface diffeomorphisms): existence of an open set $\mathcal{U}$ and a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that every $f \in \mathcal{R}$ has infinitely many sinks and/or sources.

A relevant point here is that this construction does not involve explicitly homoclinic tangencies (although such tangencies are displayed) but it is related to the generation of homoclinic classes which do not admit any dominated splitting. In fact, this construction of non-dominated homoclinic classes is the main motivation behind the results in the next sections.

In Section 4, we discuss a dichotomy *weak hyperbolicity versus coexistence phenomenon* for C^1-generic diffeomorphisms (i.e., diffeomorphisms in a residual subset of $\text{Diff}^1(M)$), see Theorem 5. This implies that every homoclinic class of a C^1-generic diffeomorphisms which does not admit any dominated splitting is contained in the closure of an infinite set of sinks or sources.

Strong forms of the coexistence phenomenon associated to special types of non-dominated homoclinic classes are discussed in Section 5, see Theorem 6. In Theorem 7 we present the main argument in these constructions: some non-dominated homoclinic classes generate periodic dynamics close to the identity (the so-called *universal dynamics*).

In Section 4, we also discuss some ingredients in the proof of Theorem 5 (as the transitions for homoclinic classes) and explain how the absence of domination generates sinks and sources in the two-dimensional case (this allows to present some of the ideas in the proofs avoiding technicalities)

Finally, in Section 6, we introduce a sort of bifurcation which generates in a natural way non-dominated dynamics (thus the coexistence phenomena): *heterodimensional tangencies*, see Theorem 8. In rough terms, in a manifold of dimension three, we consider a homoclinic class containing a saddle P whose stable manifold is two dimensional and a saddle Q whose unstable manifold is two dimensional, a heterodimensional tangency means that these two manifolds have non-transverse intersections. We belief that this bifurcation plays in the partially hyperbolic setting a role similar to the one of homoclinic tangencies in the hyperbolic context (a way for crossing the boundary of hyperbolicity) describing a natural transition from dynamics with some hyperbolicity to non-dominated dynamics. Indeed, these heterodimensional tangencies are not well understood and its bifurcation theory is an open research subject.

2 Around Smale's density conjecture: robust cycles

In [45], Smale conjectured the density of hyperbolic dynamics among diffeomorphisms. This conjecture holds in the one-dimensional settingIn higher dimensions either this conjecture does not hold (in the C^1-topology for manifolds of dimension $n \geq 3$, and in the C^r-topology, $r \geq 2$, for any dimension $n \geq 2$) or it remains open (for C^1-surface diffeomorphisms). In fact, some counter-examples of different nature were given to this conjecture. In this section, we analyze some of these counter-examples, considering first the C^1-ones. We close this section by discussing recent results about this density conjecture for C^1-surface diffeomorphisms.

2.1 Heterodimensional cycles and dominated dynamics

A natural (and the most usual) strategy for proving the hyperbolicity of a set is to begin by considering its hyperbolic periodic points. These points have a natural hyperbolic structure given by the eigenspaces. Next one checks if these hyperbolic structures fit together nicely and if they can be extended to the closure. This leads to the notion of Axiom A diffeomorphism: the periodic points of the diffeomorphism f are dense in its non-wandering set $\Omega(f)$ and this set is hyperbolic. In this Axiom A case, the *spectral decomposition* theorem ([32]) claims that the non-wandering set splits into finitely many pairwise disjoint sets $\Omega(f) = \Lambda_1(f) \cup \cdots \cup \Lambda_n(f)$, where each $\Lambda_i(f)$ is hyperbolic, transitive, and locally maximal. The sets Λ_i are the *basic sets* of the spectral decomposition of f.

The first sort of counter-examples (in the context of C^1-diffeomorphisms) to the density conjecture was initiated with the construction by Abraham-Smale in [3]. The authors considered a four dimensional manifold M^4 and constructed an open set $\mathcal{C}(M^4)$ of the space of C^1-diffeomorphisms $\mathrm{Diff}^1(M^4)$ of M^4 such that every $f \in \mathcal{C}(M^4)$ has two transitive hyperbolic sets Λ_f and Γ_f of different *index* (dimension of the unstable bundle) related by a cycle: the unstable manifold of Λ_f intersects the stable one of Σ_f and vice-versa. Moreover, one of these intersections is transverse. This dynamical configuration prevents the diffeomorphisms from being Axiom A. Otherwise, both sets should be contained in some basic set of the spectral decomposition. Noting that all the periodic points of a transitive hyperbolic set have the same index one gets a contradiction.

The Abraham-Smale's construction leads to the following definition:

Definition 2.1 (Robust heterodimensional cycles). *A diffeomorphism f has a* heterodimensional cycle *associated to the (transitive) hyperbolic sets Γ and Σ of f if:*

1. *the indices of the sets Γ and Σ are different;*
2. *the stable manifold of Γ meets the unstable manifold of Σ and the same holds for the stable manifold of Σ and the unstable manifold of Γ.*

*The heterodimensional cycle of f associated to the sets Γ and Σ is C^r-*robust *if there is a C^r-neighbourhood $\mathcal{U}$ of f such that any diffeomorphism $g \in \mathcal{U}$ has a heterodimensional cycle associated to the hyperbolic sets Γ_g and Σ_g, where Γ_g and Σ_g are the continuations of Γ and Σ for g.*

Clearly, heterodimensional cycles can only occur in manifolds of dimension greater than or equal to three. Moreover, by the Kupka-Smale genericity theorem (generically, periodic points are hyperbolic and their invariant manifolds are in general position) robust cycles cannot be associated to trivial hyperbolic sets (saddles).

In the construction in [3], the non-wandering set does not support a hyperbolic splitting but it exhibits some form of weak hyperbolicity, called *partial hyperbolicity* and defined as follows.

Definition 2.2 (Dominated and partially hyperbolic splittings).

- *An f-invariant set Λ has a* dominated splitting *if the tangent bundle $T_\Lambda M$ over Λ splits into two Df-invariant bundles E and F, $T_\Lambda M = E \oplus F$, whose fibers E_x and F_x have constant dimensions, and there exists an integer $\ell \geq 1$ such that, for every point $x \in \Lambda$ and every pair of unit vectors $u \in E_x$ and $v \in F_x$ it holds that*

$$\|Df^\ell(x)\, u\| \leq \frac{1}{2}\, \|Df^\ell(x)\, v\|.$$

  *In this case, we say that splitting is ℓ-*dominated *and that F dominates E.*

- *If the dominated splitting $T_\Lambda M = E \oplus F$ is such that one of the bundles is uniformly hyperbolic we say that it is* partially hyperbolic.

Let us mention some important properties of dominated splittings that we will use in this paper (for details see, for instance, [11, Appendix B] and for a survey on the dynamics of sets having dominated splittings see [40]).

1. (continuous dependence of the fibers) The fibers E_x and F_x of the dominated splitting depend continuously on the point $x \in \Lambda$.

2. (bounded angle) The angle between the bundles E_x and F_x, $x \in \Lambda$, is uniformly bounded away from below.

3. (extension to the closure) Suppose that $E \oplus F$ is a dominated splitting defined on an f-invariant set Λ. Then there is a dominated splitting $E' \oplus F'$ defined on the closure of Λ such that $E_x = E'_x$ and $F_x = F'_x$, for all $x \in \Lambda$.

The construction in [3] was the first one of a long list of counter-examples to Smale's conjecture for C^1-diffeomorphisms. In these constructions, the non-wandering set is not hyperbolic but it has a dominated splitting (in some cases the splitting is partially hyperbolic). First, Simon, [44], performed a similar construction to the one in [3] in the three torus. Later Shub, [43], and Mañé, [29] proved that in these constructions one can take $\Omega(f)$ being the whole manifold. Finally, [5, 12] gave new kind of counter-examples of different nature involving the notion of *blender* and *heterodimensional cycles*. We will analyze this concept and the dynamics at heterodimensional cycles in Section 3.

More recently, for manifolds M of dimension 3, in [6, 8] a different kind of open sets $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ of non-hyperbolic diffeomorphisms was constructed (these constructions can be carried out in higher dimensions straightforwardly considering normally hyperbolic sub-manifolds). The open set $\mathcal{U}$ contains a residual set $\mathcal{R}$ of diffeomorphisms f having infinitely many sinks or sources. In fact, the set $\mathcal{U}$ can be constructed having a residual subset $\mathcal{R}$ of diffeomorphisms having simultaneously infinitely many sinks, infinitely many sources, and infinitely many (non-trivial) minimal sets. Recall that an invariant set Σ is *minimal* if the orbit of any point of Σ is dense. The nature of these examples is completely different of those mentioned above. In fact, they are related to robustly non-dominated dynamics, the main subject of this note. We will discuss these constructions in Section 4 and 5. Actually, the constructions in [6, 8] were inspired and motivated by Newhouse's construction of C^2-diffeomorphisms with infinitely many sinks we will discuss in the next section.

2.2 Homoclinic tangencies and non-dominated dynamics

We mow describe the counter-example to the density conjecture given by Newhouse in the context of C^2-surface diffeomorphisms. This kind of construction was later generalized by Palis-Viana, [38], and Romero, [41], to higher dimensions. In the Newhouse's construction, the non-wandering sets of the diffeomorphisms do not admit any dominated splitting: the non-wandering set contains a sequence of hyperbolic periodic saddles P_n such that, for each n, the angle between the stable and unstable directions of P_n is less than $1/n$. Observe that if $E \oplus F$ is a dominated splitting of a surface diffeomorphism and P is a saddle then $E_P = E^s_P$ and $F_P = E^u_P$. Recalling that the angles between the bundles of a dominated splitting are bounded away from zero, one gets the claim. These constructions rely on the notions of persistent and robust homoclinic tangencies:

Definition 2.3 (Persistent and robust homoclinic tangencies). *Let f be a diffeomorphism defined on a closed manifold M.*

- *The diffeomorphism f has a C^r-*persistent homoclinic tangency* associated to a hyperbolic periodic saddle P if there exist a C^r-neighbourhood $\mathcal{U}$ of f and a dense subset $\mathcal{D}$ of $\mathcal{U}$ such that, for every diffeomorphisms $g \in \mathcal{D}$, the continuation P_g of P has a* homoclinic tangency *(i.e., the stable and unstable manifolds of P_g have some non-transverse intersection).*

- *Let Λ be a transitive hyperbolic set of f. We say that Λ has a C^r-*robust tangency *if there are a C^r-neighbourhood $\mathcal{U}$ of f and a constant $T > 0$ such that for any $g \in \mathcal{U}$ the local stable manifold of size T of Λ_g, $W^s_T(\Lambda_g)$, and the unstable manifold of size T of Λ_g, $W^u_T(\Lambda_g)$, have a tangency. Here Λ_g is the hyperbolic continuation of Λ for g.*

As in the case of heterodimensional cycles, robust homoclinic tangencies cannot be associated to trivial hyperbolic sets.

A key result about homoclinic tangencies of C^2-surface diffeomorphisms is the following:

Theorem 1 (Newhouse, [33, 34]). *Let M be a closed surface.*

- *There is an open set $\mathcal{T}$ of $\mathrm{Diff}^2(M)$ consisting of diffeomorphisms with C^2-robust tangencies.*
- *There is a residual subset $\mathcal{S}$ of $\mathcal{T}$ of diffeomorphisms with infinitely many sinks or sources.*
- *Consider a C^2-diffeomorphism f with a homoclinic tangency associated to a saddle. Then there are an open set $\mathcal{U}$ of $\mathrm{Diff}^2(M)$ whose closure contains f and a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that every diffeomorphism in $\mathcal{R}$ has infinitely many sinks or sources.*

We say that a diffeomorphism f having a C^k-neighbourhood $\mathcal{U}$ such that there is a residual subset of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks or sources satisfies the (Newhouse's) *coexistence phenomenon.*

For a wide discussion and a exposition of these results we refer to [37]. For a survey about features associated to homoclinic tangencies, see [36] and [40, Chapter 3].

We now discuss briefly the first two items of Theorem 1. The proof of this theorem relies on the notions of homoclinic tangency associated to a saddle and of *thickness of a Cantor set.* We begin by defining precisely thickness and stating the so-called Gap Lemma, which relates the sum of the thickness of a pair of Cantor sets of $\mathbb{R}$ and the intersection (also relative position) of these sets.

The thickness of a Cantor set of $\mathbb{R}$ is a fractal dimension, defined in the same spirit as Hausdorff dimension and limit capacity. It describes the relation between the lengths of the intervals removed in the construction of a Cantor set and the lengths of the two remaining adjacent connected components. More precisely, consider a Cantor set $K \subset \mathbb{R}$ whose convex hull is the interval $[a, b]$. A presentation of the Cantor set K is an enumeration of the connected components $\{G_i\}_{i\in\mathbb{N}}$ of $[a, b] \setminus K$. We call these components *gaps* of K. Let $F_0 = [a, b]$ and, for $i \geq 1$, we let $F_i = F_{i-i} \setminus G_i$. Note that each set F_i is the union of $(i + 1)$ pairwise disjoint closed intervals. Let B^r_i and B^ℓ_i the connected components of F_i intersecting the boundary of G_i. We call these intervals *bridges* of G_i. The *thickness of the presentation* $\{G_i\}$ of K is

$$\tau(K, \{G_i\}) = \inf_{i\geq 1} \left\{ \min \frac{|B^r_i|}{|G_i|}, \frac{|B^\ell_i|}{|G_i|} \right\}.$$

Finally, the thickness of K, $\tau(K)$, is the supremum of $\tau(K, \{G_i\})$ taken over all presentations $\{G_i\}$ of K. In our context, the importance of thickness is given by the next lemma.

Lemma 2.4 (Gap Lemma, [33, 37]). *Let K_1 and K_2 be Cantor sets of $\mathbb{R}$ with $\tau(K_1)\cdot\tau(K_2) > 1$ and such that their convex hulls are non disjoint. Then either $K_1 \cap K_2 \neq \emptyset$ or K_1 is contained in a gap of K_2 or vice-versa.*

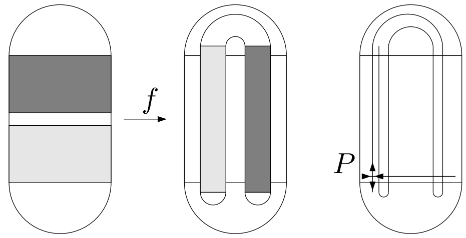

Figure 1: A thick horseshoe

We are now ready to sketch the construction of diffeomorphims having robust tangencies. Consider, for simplicity, a diffeomorphism f having a linear horseshoe Λ defined on a rectangle of $\mathbb{R}^2$ as in Figure 1 (the vertical direction is expanding and the horizontal one is contracting). Consider the saddle $P = (0, 0)$ and the Cantor sets defined by

$$\Lambda^u = (\{0\} \times [0, 1]) \cap \Lambda = W^u_{loc}(P) \cap \Lambda \quad \text{and} \quad \Lambda^s = ([0, 1] \times \{0\}) \cap \Lambda = W^s_{loc}(P) \cap \Lambda.$$

These Cantor sets are *dynamically defined* (for the precise definition and properties see [37, Chapter 4.1]). More precisely, suppose that the rectangle is $[0, 1]^2$ and that the expansion and contraction rates are $\sigma > 1$ and $0 < \lambda < 1$, then the Cantor Λ^u is the set of points $(0, x)$, $x \in [0, 1]$, such that $\Phi^i_\sigma(x) \notin (1/\sigma, 1 - 1/\sigma)$ for all $i \geq 0$, here Φ_σ is the affine map in Figure 2.

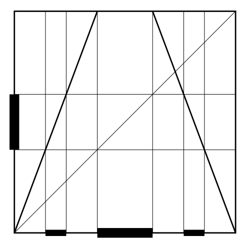

Figure 2: The dynamically defined set Λ^u

We say that the interval $G = (1/\sigma, 1-1/\sigma)$ is the first generation gap of the Cantor set Λ^u. The sets $[0, 1/\sigma]$ and $[1 - 1/\sigma, 1]$ are the initial bridges of Λ^u. Further gaps of the Cantor sets are the pre-images of G by Φ_σ (in Figure 2 are depicted the gaps of second generation). The bridges are the connected components of $[0, 1]$ obtained after removing the corresponding gaps. In this linear case, due to the affinity of the Cantor sets, the thickness of Λ^u and Λ^s can be easily calculated,

$$\tau(\Lambda^u) = \frac{1/\sigma}{1 - 2/\sigma} \quad \text{and} \quad \tau(\Lambda^u) = \frac{\lambda}{1 - 2\lambda}.$$

This means that if λ and σ are both close to one then the thickness $\tau(\Lambda^u)$ and $\tau(\Lambda^s)$ are both large. Thus the *thickness of* Λ, defined as the sum $\tau(\Lambda) = \tau(\Lambda^s) + \tau(\Lambda^u)$, is large (in particular, bigger than one). We say that Λ is a *thick* horseshoe if $\tau(\Lambda) > 1$.

Observe that the local stable manifolds of Λ are horizontal segments and the local unstable manifolds are vertical ones. We modify the dynamics of f outside the horseshoe Λ to create a point z of tangency between the stable and the unstable manifolds of the saddle P. After this perturbation, locally at the point $z = (z_1, z_2)$, the stable manifold of P is a horizontal segment γ^s and the unstable one is a parabola γ^u. The perturbation is depicted in Figure 3.

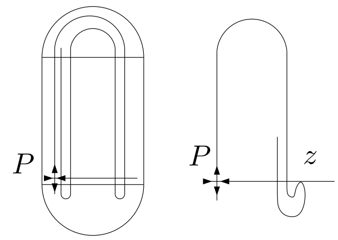

Figure 3: A horseshoe with a tangency

The next step is to unfold the tangency, considering an arc of diffeomorphisms $(f_t)_{t\geq 0}$, $f_0 = f$, preserving the horseshoe Λ and the local stable manifold of P containing z, and translating the parabolic segment γ_u in the vertical direction by the vector $(0, t)$ (that is, the parabolic curve $\gamma^u_t = \{(x_1, x_2 + t), (x_1, x_2) \in \gamma^u\}$ is contained in the unstable manifold of P for f_t). See Figure 4.

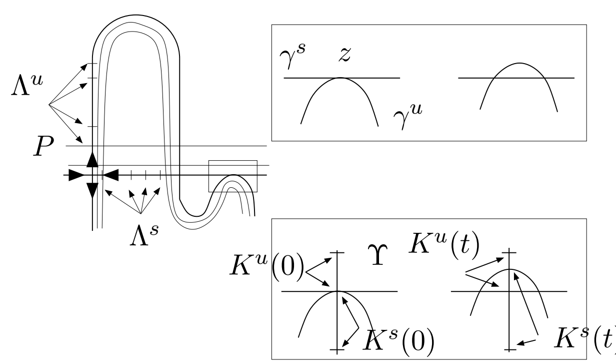

Figure 4: Unfolding of the tangency

We now explain how the persistence of tangencies arises. Observe that, for each t, the horseshoe Λ is the product of the Cantor sets $\Lambda^s = \Lambda \cap W^s_{loc}(P, f_t)$ and $\Lambda^u = \Lambda \cap W^u_{loc}(P, f_t)$ (in fact, in this case these Cantor sets do not depend on t). Consider the curve $\Upsilon = \{z_1\} \times [-\epsilon, +\epsilon]$ containing the tangency point z and, for each $t \geq 0$, the Cantor sets

$$K^u(t) = W^s(\Lambda^u, f_t) \cap \Upsilon = W^s(\Lambda, f_t) \cap \Upsilon \quad \text{and} \quad K^s(t) = K^s = W^u(\Lambda^s, f_t) \cap \Upsilon = W^s(\Lambda, f_t) \cap \Upsilon.$$

Then

$$K^u(0) \subset \Upsilon^+ = \{z_1\} \times [0, +\epsilon] \quad \text{and} \quad K^s(0) \subset \Upsilon^- = \{z_1\} \times [-\epsilon, 0].$$

The same construction for parameters $t > 0$ gives

$$(z_1, 0) \in K^u(t) \subset \{z_1\} \times [0, +\epsilon] \quad \text{and} \quad (z_1, t) \in K^s(t) \subset \{z_1\} \times [-\epsilon, t].$$

One now has the sets $K^s(t)$ and $K^u(t)$ are *linked,* meaning that their convex hulls have non-empty intersection: the point $(z_1, 0)$ belongs to the convex hulls of $K^s(t)$ and $K^u(t)$. A key point now, involving the fact that the thickness is invariant by bi-Lipschitz transformations, is that the thickness of the Cantor sets $K^u(t)$ and $K^s(t)$ are equal to the thickness of Λ^u and Λ^s. Thus if we start our construction with a linear horseshoe with $\tau(\Lambda^s) + \tau(\Lambda^u) > 1$ (one can assume this hypothesis by the comments above), noting that the sets $K^u(t)$ and $K^s(t)$ are linked, the Gap Lemma implies that $K^u(t) \cap K^s(t) \neq$ for all small $t \geq 0$.

The construction ends by observing that an intersection between $K^u(t)$ and $K^s(t)$ corresponds to a tangency of Λ for f_t.

It remains to explain why the previous construction is C^2-robust: consider $t > 0$ such that $K^u(t) \cap K^s(t) \neq \emptyset$, then there is a C^2-neighborhood $\mathcal{V}$ of f_t in $\mathrm{Diff}^2(M)$ such that for every $g \in \mathcal{V}$ the continuation Λ_g of Λ for g has a tangency (i.e., $W^s(\Lambda_g, g)$ and $W^u(\Lambda_g, g)$ have some non-transverse intersection). The key point here is that the thickness of the Cantor sets involved in the construction depends continuously (see [37] for the details). That is the maps

$$\tau^i \colon \mathcal{V} \to \mathbb{R}, \quad g \mapsto \tau^i(g) = \tau(W^s_{loc}(P_g, g) \cap \Lambda_g), \quad i = s, u,$$

are continuous (here P_g and Λ_g are the continuation of the saddle P and the hyperbolic set Λ for g). This means that we can repeat the procedure above for the diffeomorphism g close to f_t, obtaining a tangency between $W^s(\Lambda_g, g)$ and $W^u(\Lambda_g, g)$. One also has that, by construction, these tangencies occur in segments of uniformly bounded size of the local stable and local unstable manifolds of the hyperbolic sets.

The previous arguments are key ingredient in Newhouse's construction of locally residual sets of diffeomorphisms having infinitely many sinks or sources in Theorem 1. The other crucial ingredient is that every surface diffeomorphism f with a homoclinic tangency associated (say) to a dissipative saddle P (i.e., the absolute value of the product of the eigenvalues of the saddle is less than one) is C^2-approximated by diffeomorphism g having a periodic sink Q whose orbit is arbitrarily close to the one of P, see [33]. This assertion can also be proved observing that the unfolding of a homoclinic tangency generates horseshoes, and that this generation accomplishes a cascade of period doubling bifurcations and sinks (see [47] and [37, Chapter 3.3]). An intuitive explanation of the creation of these sinks is the following: (after a re-scaling) the restriction of the returns of f_t close to the tangency are close to the one-dimensional quadratic family, and this family generates sinks. For details, see [37, Chapter 3.4].

Consider now an open set $\mathcal{T}$ of diffeomorphisms f having robust tangencies as the set $\mathcal{V}$ above (i.e., there is a continuous map Λ defined on $\mathcal{V}$ which associates to each $g \in \mathcal{V}$ a transitive hyperbolic set Λ_g such that $W^s(\Lambda_g, g)$ and $W^u(\Lambda_g, g)$ have some non-transverse intersection). First, one checks that there is a dense subset $\mathcal{D}$ of $\mathcal{T}$ of diffeomorphisms g with homoclinic tangencies associated to some saddle of Λ_g (assume that these saddles are dissipative). To get a residual subset $\mathcal{S}$ of $\mathcal{T}$ of diffeomorphisms with infinitely many sinks, note first that to have a sink is an open property. Thus the set $\mathcal{S}_k$ of diffeomorphisms with k (different) sinks is open. To prove Theorem 1 it suffices to check that each $\mathcal{S}_k$ is dense in $\mathcal{T}$ and to take $\mathcal{S} = \cap \mathcal{S}_k$. By construction, the set $\mathcal{S}$ is residual in $\mathcal{T}$ and every diffeomorphism f of $\mathcal{S}$ has infinitely many sinks. The density of the sets $\mathcal{S}_k$ follows inductively, it is enough to recall that, by the comments above, every diffeomorphism in the dense set $\mathcal{D}$ can be approximated by a diffeomorphism with a (new) sink of arbitrarily large period. The density of $\mathcal{D}$ in $\mathcal{U}$ gives the density of $\mathcal{S}_1$. The inductive pattern is now clear: to construct $\mathcal{S}_{k+1}$ from $\mathcal{S}_k$ one produces new homoclinic tangencies (this is possible by the density of $\mathcal{D}$) and by unfolding the tangency one gets a new sink preserving the previous ones.

2.3 Smale's conjecture for C^1 surface diffeomorphisms

We next discuss the density conjecture of Smale for C^1-surface diffeomorphisms. We first recall that, in this case, this conjecture still remains open. In view of the previous results, one is tempted to disprove this conjecture by constructing a C^1-open set of diffeomorphisms with robust tangencies.

Before trying to do that, we observe that the ingredients of Newhouse's construction are typically C^2 (bounded distortion and fractal dimensions). Moreover, in [46] Ures proved that hyperbolic sets of C^1-generic diffeomorphisms have zero thickness. This is a first indication that the constructions above cannot be carried directly to the C^1-topology. Moreover, Moreira recently announced that dynamically defined Cantor sets defined by C^1-maps cannot have robust intersections, see [31]. Since dynamically defined Cantor sets are archetypes of hyperbolic sets of surface diffeomorphisms (obtained by *quotienting* along the invariant manifolds) this result can be viewed as a strong indication of the fact that, in the C^1-topology and for surface diffeomorphisms, there are no robust homoclinic tangencies. Unfortunately, to prove the density conjecture it is not enough to see that tangencies associated to hyperbolic sets never are robust. We now discuss this point. In [1] is proved that there are two sort of difficulties for proving this conjecture.

Theorem 2 (Thrichotomy for Smale's conjecture, [1]). *Let M be a closed surface. There are three disjoint open sets $\mathcal{H}$, $\mathcal{P}$, and $\mathcal{W}$ with*

$$\overline{\mathcal{H} \cup \mathcal{P} \cup \mathcal{W}} = \mathrm{Diff}^1(M)$$

such that:

- *$\mathcal{H}$ is the set of diffeomorphisms which satisfy the Axiom A and the no-cycle condition;*
- *$\mathcal{P}$ is the set of diffeomorphisms admitting a persistent homoclinic tangency associated to some hyperbolic periodic saddle;*
- *$\mathcal{W}$ contains a residual subset $\mathcal{GW}$ such that every $f \in \mathcal{GW}$ has infinitely many homoclinic classes, all of them being hyperbolic basic sets.*

Recall that an Axiom A diffeomorphism f has a *two-cycle* (or shortly, a *cycle*) if there are two basic sets $\Lambda_i(f)$ and $\Lambda_k(f)$ of the spectral decomposition of its non-wandering set $\Omega(f)$ such that $W^s(\Lambda_i(f)) \cap W^u(\Lambda_k(f)) \neq \emptyset$ and $W^u(\Lambda_i(f)) \cap W^s(\Lambda_k(f)) \neq \emptyset$ [2].

Let us also recall that the *homoclinic class* of a saddle P of a diffeomorphism f, denoted by $H(P, f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit of P. A homoclinic class can be also (equivalently) defined as the closure of the set of (hyperbolic) saddles Q *homoclinically related* to P (the stable manifold of the orbit of Q transversely meets the unstable one of the orbit of P and vice-versa). In the Axiom A case, the (non-trivial) basic sets in the spectral decomposition are homoclinic classes.

A natural question is whether the diffeomorphisms $f \in \mathcal{P}$ with persistent homoclinic tangencies have robust tangencies. Consider the following two disjoint C^1-open sets $\mathcal{P}_{rob}$ and $\mathcal{P}_\infty$ of $\mathcal{P}$:

- the set $\mathcal{P}_{rob}$ consists of diffeomorphisms admitting a robust tangency associated to some hyperbolic set;

[2] The general definition of cycle involves n basic sets $\Lambda_1, \dots, \Lambda_n$ such that, for each $i \in \{1, \dots, n\}$, it holds that $W^u(\Lambda_i(f)) \cap W^s(\Lambda_{i+1}(f)) \neq \emptyset$, where $n + 1 \equiv 1$

- the set $\mathcal{P}_\infty$ has a residual subset $\mathcal{GP}_\infty$ such that for every $g \in \mathcal{GP}_\infty$ and for any hyperbolic set Λ of g, the invariant manifolds of Λ are transverse (even though g presents persistent tangencies in the sense of Definition 2.3).

[1, Proposition 1.4] claims that the union of the sets $\mathcal{P}_{rob}$ and $\mathcal{P}_\infty$ is dense in $\mathcal{P}$. The previous results show that besides the existence of C^1-robust tangencies two other different obstructions can appear for a diffeomorphism f satisfying the Axiom A and no cycles property: the *creation of tangencies outside the homoclinic class* (corresponding to diffeomorphisms in $\mathcal{P}_\infty$) or the simultaneous existence (with some persistence) of diffeomorphisms with infinitely many different hyperbolic homoclinic classes (corresponding to diffeomorphisms in $\mathcal{GW}$).

Having in mind these results, [1] proposes to split the Smale's conjecture into the following three different conjectures:

Conjecture 1 (Splitting Smale's conjecture for C^1-surface diffeomorphisms, [1]).

1. *Every generic diffeomorphism of a compact surface whose homoclinic classes are all hyperbolic satisfies Axiom A.*
2. (No robust tangencies) *Let Λ be a hyperbolic set of a diffeomorphism f of a compact surface. Then, for any $L > 0$, there is a C^1-perturbation g of f such that the local invariant manifolds of size L of the hyperbolic set Λ_g (Λ_g is the continuation Λ_g of Λ for g) are transverse.*
3. (Persistent tangencies imply robust tangencies) *Any diffeomorphism f with persistent tangencies associated to some saddle may be C^1-approximated by diffeomorphisms with robust tangencies.*

By Theorem 2, a positive answer to these conjectures implies the Smale's conjecture for surface diffeomorphisms.

3 Heterodimensional cycles

In this section, we discuss the dynamics generated by heterodimensional cycles. We see that the creation of such cycles is a natural mechanism for the generation of robustly non-hyperbolic transitive sets (in fact, homoclinic classes). The assumptions here are not minimal, but we point out that the dynamical behaviour depicted here is typical of a large class of heterodimensional cycles, see [19, 7]. For a review on heterodimensional cycles, see [11, Chapter 6].

We consider diffeomorphisms f with a cycle associated to hyperbolic periodic saddles P and Q (for simplicity, assume that these saddles are fixed points) such that

- (co-index 1) the saddles P and Q have indices p and $q = p + 1$, respectively,
- (general position: transverse intersection) the manifolds $W^s(P, f)$ and $W^u(Q, f)$ (of dimension $n-p$ and $p+1$, n is the dimension of the ambient) meet transversely and this intersection contains curves, and
- (general position: quasi-transverse intersection) the manifolds $W^u(P, f)$ and $W^s(Q, f)$ (of dimension p and $n - p - 1$) intersects throughout the orbit of some point x and this intersection is quasi-transverse: $T_xW^s(Q, f) \cap T_xW^u(P, f) = \{\bar{0}\}$ or, equivalently, $T_xW^s(Q, f) + T_xW^u(P, f) = T_xW^s(Q, f) \oplus T_xW^u(P, f)$ and this sum has dimension $n - 1$.

The cycles above are called *co-index one cycles.* Note that in dimension three every heterodimensional cycle has co-index one. A heterodimensional cycle is depicted in Figure 5.

A result which illustrates heterodimensional cycles as a natural mechanism for C^1-robust non-hyperbolic dynamics is the following:

Theorem 3 ([7]). *Let f be a C^1-diffeomorphism having a co-index one cycle associated to a pair of hyperbolic saddles. Then there are diffeomorphisms arbitrarily C^1-close to f having robust (heterodimensional) cycles.*

This theorem generalizes some previous results in [19]. The proof of Theorem 3 involves recent results on C^1-generic dynamics and the idea of *blender* generated by a heterodimensional cycle, developed in previous papers, [5, 19]. For a rough explanation of blenders, we need to review some previous results about heterodimensional cycles.

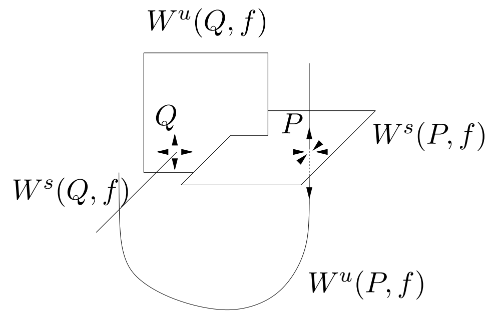

Figure 5: A heterodimensional cycle

We first see a theorem claiming that, after unfolding a heterodimensional cycle, the homoclinic classes of the saddles P and Q in the cycle often explode and become intermingled (non-empty intersection). Consider parametrized families $(f_t)_{t\in[-1,1]}$ of diffeomorphisms *unfolding* the cycle: we have $f = f_0$ and there are disks $K_t^u \subset W^u(P_t, f_t)$ and $K_t^s \subset W^s(Q_t, f_t)$ (of the same dimension as $W^u(P_t, f_t)$ and $W^s(Q_t, f_t)$), depending continuously on t, such that $K_0^u \cap K_0^s = \{x\}$, where x is a point of quasi-transverse intersection, and the distance between K_t^s and K_t^u increases with positive velocity when $|t|$ increases. Here P_t and Q_t denote the continuations for f_t of P and Q.

Theorem 4 ([17, 14, 15]). *There is a non-empty open set $\mathcal{A}$ of parametrized C^∞-families of diffeomorphisms $(f_t)_{t\in[-1,1]}$ unfolding a heterodimensional cycle of $f = f_0$, such that, for all small positive t,*

1. *the transverse intersection between $W^s(P_t)$ and $W^u(Q_t)$ is contained in the homoclinic class of Q_t;*

2. *the homoclinic class of P_t is contained in the homoclinic class of Q_t.*

The simplest case of diffeomorphisms with heterodimensional cycles in the theorem are obtained as follows. Consider a diffeomorphism f defined on a three manifold (in this case, the indices of the saddles P and Q in the cycle are one and two, respectively) such that the eigenvalues of $Df(P)$ and $Df(Q)$ verify $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ and $0 < \beta_s < 1 < \beta_c < \beta_u$, respectively. Moreover, the transverse intersection between the two dimensional manifolds $W^s(P, f)$ and $W^u(Q, f)$ contains a curve γ (named *connection*) with endpoints P and Q. We also assume that the curve γ is simultaneously transverse to the strong stable foliation of $W^s(P, f)$ and to the strong unstable foliation of $W^u(Q, f)$ (the unique f-invariant one-dimensional foliations of $W^s(P, f)$ and $W^u(Q, f)$ whose leaves through P and Q are tangent to the eigenspaces of λ_s and β_u). In this case, we say that the cycle is *non-critical.*

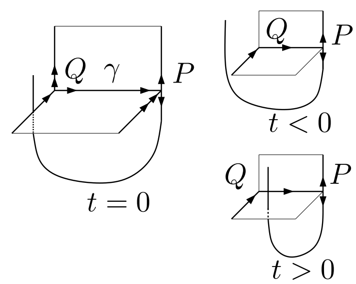

Figure 6: A heterodimensional cycle and its unfolding

In this simplest case, a substantial part of the dynamics after the unfolding of the cycle is essentially determined by the restriction of the bifurcating diffeomorphism f to the connection curve γ. This claim is suggested by the series of papers [14, 18, 20], where the dynamics in the sequel of the bifurcation (creation of the cycle) is reduced to the analysis of a system of iterated functions generated by f and a translation. For an expository explanation of this dynamics see [11, Chapter 6] and [21]. This one-parameter system of iterated functions plays a role somewhat similar to the one of the quadratic family for homoclinic bifurcations, see [37, Chapter 3.4].

Finally, when the distortion of the restriction of f to the connection γ is small (in fact, the definition of the open set $\mathcal{A}$ in Theorem 4 involves this property) the system of iterated functions describing the dynamics is expanding. This is the key behind the *distinctive property* of heterodimensional cycles, which is also the key for proving that after unfolding the cycle the homoclinic classes of P_t and Q_t are intermingled: for every small $t > 0$, the homoclinic class of Q_t contains the one of P_t.

Distinctive property: *For every $t > 0$ and every 2-disk Δ transverse to $W^s(P, f_t)$, the stable manifold $W^s(Q, f_t)$ of Q_t intersects transversely the disk Δ. Thus the closure of the one-dimensional stable manifold $W^s(Q, f_t)$ contains the two-dimensional stable manifold $W^s(P_t, f_t)$ of P_t.*

Heuristically, this means that the stable and unstable manifolds of Q_t have both (topological) dimension two. Therefore the saddle Q_t behaves simultaneously as a point of indices one and two. This is the main step for proving that the homoclinic classes of P_t is contained in the homoclinic class of Q_t.

The constructions in [14] of homoclinic classes containing (robustly) saddles having different indices was generalized and systematized in [5], where the notion of *blender* was introduced. Roughly, a blender is a topological *plug* depending only on semi-local properties, which guarantees that the closure of the one dimensional stable manifold of a saddle of index 2 contains (C^1-robustly) the two dimensional stable manifold of a saddle of index 2. For an expository construction of blenders, see [11, Chapter 6.2].

We observe that topological dimension property of blenders (the dimension of the unstable manifold of the blender is greater than its index) is a C^1-robust property. This property plays

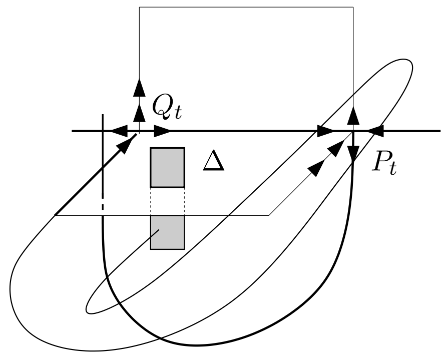

Figure 7: The distinctive intersection property

a role similar to the thick hyperbolic sets (recall Section 2.2) in the construction of C^1-robust heterodimensional cycles in Theorem 3.

An important point in Theorem 4 is that the only assumption on the dynamics of the bifurcating diffeomorphism f involved in the proof (besides the type of geometry of the curve $\gamma \subset W^s(P,f) \cap W^u(Q,f)$) is that the restriction of f to the connection γ has small distortion. These conditions are compatible with other hypotheses on the global dynamics of f. For instance, the saddle P can be homoclinically related to a saddle P' of index one such the contracting eigenvalues of $Df(P')$ are conjugate and non-real. A similar situation can occur for Q. This dynamical configuration is depicted in Figure 8.

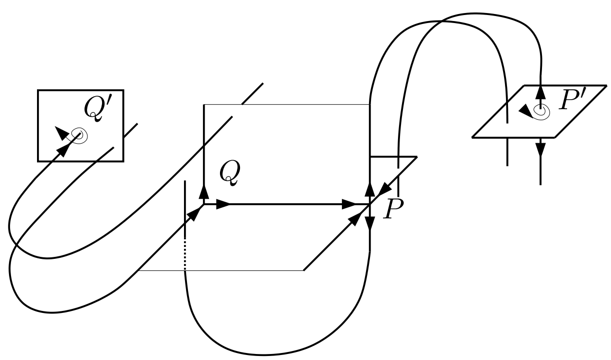

Figure 8: Generation of the coexistence phenomenon

In fact, assuming either the existence of some *sectionally dissipative* saddle (the modulus of the product of any pair of multipliers is less than one) homoclinically related to P or the existence of some sectionally expansive saddle homoclinically related to Q, [6] proved that dynamical configuration in Figure 8 generates the C^1-coexistence phenomenon: there is a C^1-open set $\mathcal{C}$ and a residual subset $\mathcal{R}$ of $\mathcal{C}$ such that every $g \in \mathcal{R}$ has infinitively many sinks/sources.

Observe that, under the hypotheses above, one has a C^1-open set $\mathcal{C}$ consisting of diffeomorphism g such that

$$H(P'_g, g) = H(P_g, g) \subset H(Q_g, g) = H(Q'_g, g).$$

We need the following lemma about the creation of intersections between invariant manifolds of saddles in a transitive set:

Lemma 3.1 (Consequence of the Connecting Lemma, Hayashi, [25]). *Let Σ be a transitive set of a diffeomorphisms f containing a pair of saddles A and B. Then there is g arbitrarily C^1 close to f such that $W^s(A_g, g) \cap W^u(B_g, g) \neq \emptyset$.*

Assume, for instance, that the saddle P_f is sectionally dissipative. Applying first Lemma 3.1 to the transitive set $\Sigma_f = H(Q_f, f)$ and the saddles P_f and $Q' \in \Sigma_f$, we obtain a diffeomorphism g close to f such that $W^s(P_g, g)$ and $W^u(Q'_g, g)$ have some transverse intersection. A new application of the lemma gives that $W^u(P_g, g)$ and $W^s(Q'_g, g)$ have non-empty intersection. This provides a dense subset $\mathcal{D}$ of $\mathcal{C}$ of diffeomorphisms g with heterodimensional cycles associated to P_g and Q'_g.

Using the cycle associated to P_g and Q'_g, one has that the stable manifold of P_g spirals around the stable manifold of Q'_g. This allows us to get (after a perturbation) a tangency associated to P_g. As in the case of surface diffeomorphisms, we see that such a tangency generates (by a C^1-perturbation) a sink. This is done noting that (after a perturbation) one can assume that such a tangency occurs in a normally contracting surface, thus one can think of this homoclinic tangency as a two-dimensional one. We now can argue as in Section 2.2 to get a sink.

The previous arguments imply that the diffeomorphisms h of $\mathcal{C}$ having one sink form an open and dense subset of $\mathcal{C}$. We can now proceed inductively (exactly as in Section 2.2) proving that the set $\mathcal{S}_k$ of diffeomorphisms of $\mathcal{C}$ having k sinks contains an open and dense subset of $\mathcal{C}$. Now it is enough to consider the intersection $\mathcal{R} = \cap_k \mathcal{S}_k$, which is a residual subset of $\mathcal{C}$ of diffeomorphisms with infinitely many sinks.

In the next section, we will see that the dynamical configuration in Figure 8 is an archetypal example of robustly non-dominated homoclinic class. We will deduce some consequences from this lack of domination.

4 A dichotomy for homoclinic classes of C^1-diffeomorphisms: weak hyperbolicity or Newhouse's coexistence phenomenon

In the previous section, given a three dimensional manifold M we constructed a C^1-open set $\mathcal{C}$ of diffeomorphisms with (fixed) saddles P'_f and Q'_f depending continuously on f such that:

N1) the index of P'_f is one and $Df(P'_f)$ has a pair of non-real contracting eigenvalues,

N2) the index of Q'_f is two and $Df(Q'_f)$ has a pair of non-real expanding eigenvalues,

N3) the homoclinic class $H(P'_f, f)$ of P'_f is contained in the homoclinic class $H(Q'_f, f)$ of Q'_f.

Assuming sectional dissipativeness or expansiveness of these saddles, we got a residual subset $\mathcal{R}$ of $\mathcal{C}$ of diffeomorphisms with infinitely many sinks or sources.

The homoclinic classes satisfying (N1)–(N3) are examples of robustly non-dominated homoclinic classes: for every $g \in \mathcal{C}$, the homoclinic class $\Sigma_g = H(Q'_g, g)$ does not admit any dominated splitting. Otherwise, assume by contradiction that $E \oplus F$ is a dominated splitting of Σ_g such that (for instance) E is one dimensional. Consider the stable bundle $E^s_{P'_g}$ of P'_g (the eigenspace associated to the two contracting eigenvalues). Then, necessarily (this follows from the domination of the splitting) the bundle E is contained in $E^s_{P'_g}$, thus $Dg(P'_g)$ leaves invariant a one dimensional direction of $E^s_{P'_g}$, which is incompatible with the fact that $Dg(P'_g)$ has a pair of non-real contracting eigenvalues. The contradiction follows similarly when the bundle F is one dimensional by considering the saddle Q'_g and its unstable direction.

In this section, we discuss the following result which generalizes the construction before:

Theorem 5 (C^1-generic dichotomy for homoclinic classes, [9]). *Let M be a closed manifold. There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that, for every saddle P of f, the homoclinic class $H(P,f)$ of P satisfies the following dichotomy;*

- (weak hyperbolicity) *either $H(P,f)$ has a dominated splitting,*
- (coexistence phenomenon) *or $H(P,f)$ is contained in the closure of an infinite set of sinks or sources of f.*

In this theorem and in the weak hyperbolic case, if M is a surface then the homoclinic class is hyperbolic (this was proved by Mañé in [30]). When the dimension n of the manifold is three, then the splitting is *partially hyperbolic* (i.e., the one-dimensional bundle of the dominated splitting is either uniformly contracting or uniformly expanding). In higher dimensions, $n \geq 4$, there are examples of homoclinic classes which can not be approximated by diffeomorphisms having sinks or sources whose weak hyperbolic splitting is only dominated, see [12]. Finally, as we discuss in the previous section, it is an open question whether for C^1-surface diffeomorphisms the coexistence phenomenon can be eliminated.

4.1 Transitions for homoclinic classes

Recall that the homoclinic class $H(P,f)$ of the saddle P can be defined as the closure of the set of saddles Q which are homoclinically related to P (i.e., the stable manifold of the orbit of P transversely meets the unstable manifold of the orbit of Q and the same holds for the stable manifold of the orbit of Q and the unstable manifold of the orbit of P). Noting that two saddles which are homoclinically related have the same index, one has that the set of saddles of the same index as P is dense in the whole homoclinic class of P. Using the saddles homoclinically related to P one proves that a homoclinic class satisfies the following *transition property.*

The very rough idea of the transition property is the following. Consider two saddles P and Q of the same index which are homoclinically related. These saddles are accumulated by other hyperbolic periodic orbits (of the same index) whose orbits spend an arbitrarily large time nearby P, thereafter nearby Q, and so on. In fact, the existence of Markov partitions gives that, fixing any finite sequence of times, there is a periodic orbit expending the times of the sequence alternately close to P and Q, respectively. Moreover, the transition time (between a neighbourhood of P to neighbourhood of Q and vice-versa) of this orbit can be chosen bounded. This property will allow us to scatter in the whole homoclinic class of P some properties of the periodic points Q homoclinically related to P. The notion of transition translates this property into the language of linear systems, leading to the concept of *linear system with transitions.*

For the precise definitions, we refer to [9, Section 1]. We only explain these transitions with an example and some applications. Let Σ_P the set of saddles of f homoclinically related to the saddle P. Note that $\Sigma_P \subset H(P,f)$. We consider a periodic linear system $(\Sigma_P, f, \mathcal{A})$ of matrices (for each

$x \in \Sigma_P$, $A(x)$ is a matrix in $GL(n, \mathbb{R})$, n is the dimension of the ambient). To each point $x \in \Sigma_P$ of period $\pi(x)$ we associate a word

$$[M]_A(x) = (A(f^{\pi(x)-1}(x)), \dots, A(x));$$

having $\pi(x)$ letters in $GL(n, \mathbb{R})$. The matrix $M_A(x)$ is the product of the letters of the word $[M]_A(x)$. The interesting case for us occurs taking $A(x) = Df(x)$. In this case, $M_A(x)$ is a matrix of $Df^{\pi(x)}(x)$.

Consider now saddles $x_1, \dots, x_k$ homoclinically related to P. For each pair (i, j) there is a transition $f^{n_{i,j}}$ given by the dynamics of f from some neighbourhood of x_i to some neighbourhood of x_j. Let $[t_{i,j}]$ be the matrix of derivative of $f^{n_{i,j}}$ in local coordinates. Select now a list of large numbers $\alpha_1, \dots, \alpha_k$ and consider the product of matrices

$$[M_B] = [t_{m,1}]\, [M_A(x_m)]^{\alpha_m}\, [t_{m-1,m}]\, [M_A(x_{m-1})]^{\alpha_{m-1}} \cdots [t_{1,2}]\, [M_A(x_1)]^{\alpha_1}.$$

Then there is a periodic point y homoclinically related to P whose matrix $M_A(y)$ is arbitrarily close to $[M_B]$. The orbit of the periodic point y expends α_1 iterates close to x_1, then it goes to a neighbourhood of x_2 (this transition just involves a bounded number of iterations by f), it spends α_2 iterates close to x_2, and so on. The orbit of y is depicted in Figure 9. In practical terms, this means that one can multiply matrices corresponding to derivatives at different points, obtaining a good approximation of the matrix of some saddle of the class.

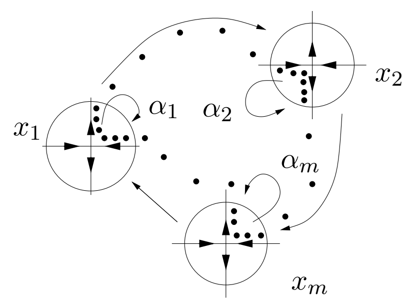

Figure 9: Transitions

The previous discussion can be summarized as follows: every non-trivial homoclinic class $H(P, f)$ has associated a periodic linear system with transitions: the basis Σ_P is the set of saddles homoclinically related to P and the linear maps are given by the derivatives of f, see [9, Lemma 1.9]. The case of transitions associated to saddles having different indices is analyzed in [10, Section 3.1].

An important consequence of the existence of transitions for homoclinic classes is the following. Given a homoclinic class $H(P, f)$, let $\mathrm{Per}_{\mathbb{R}}(P)$ be the subset of the saddles Q homoclinically related to P (thus $\mathrm{Per}_{\mathbb{R}}(P) \subset H(P, f)$) such that the eigenvalues of $Df^{\pi(Q)}(Q)$ are all real and positive and have multiplicity one. Then there is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{R}$ and for every saddle P of f whose homoclinic class is not trivial, the set $\mathrm{Per}_{\mathbb{R}}(P)$ is dense in $H(P, f)$ (this is a dynamical reformulation of [9, Lemma 4.16]). By the property *extension to the closure* of dominated splittings, given $f \in \mathcal{R}$, to prove that the homoclinic class $H(P, f)$ has a dominated splitting it is enough to see that there is a dominated splitting over $\mathrm{Per}_{\mathbb{R}}(P)$. This allows us to consider the linear periodic system induced by the derivative of f on the set $\mathrm{Per}_{\mathbb{R}}(P)$. The advantage of this sub-system is that the expression of their linear maps are quite simple (using the terminology in [9], one gets a *diagonalizable linear system*). Moreover, one also has a first natural candidate for a dominated splitting over $\mathrm{Per}_{\mathbb{R}}(P)$:

$$T_{\mathrm{Per}_{\mathbb{R}}(P)}M = H_1 \oplus H_2 \oplus \cdots \oplus H_n,$$

where n is the dimension of the ambient manifold and $H_1, \ldots, H_n$ are the eigenspaces of the saddles (the eigenvalue associated to H_{i+1} is greater than the one associated to H_i). Of course, in general this splitting fails to be dominated, but one hopes to get a dominated splitting after grouping these bundles in a suitable way.

The following holds, let

$$E_i = \oplus_{k=1}^{i} H_k \quad \text{and} \quad F_i = \oplus_{k=i+1}^{n} H_k,$$

then either $T_{\mathrm{Per}_{\mathbb{R}}(P)}M = E_i \oplus F_i$ is a dominated splitting or there is a perturbation A of the derivative of f throughout the orbit of some $Q \in \mathrm{Per}_{\mathbb{R}}(P)$ such that $M_A(Q)$ has a non-real eigenvalue of *rank* $(i, i+1)$ (i.e., ordering increasingly in modulus the eigenvalues of $M_A(Q)$, $|\lambda_1| \leq |\lambda_2| \cdots \leq |\lambda_n|$, one has that λ_i and λ_{i+1} are conjugate and non-real).

This implies that if for every i the splitting $T_{\mathrm{Per}_{\mathbb{R}}(P)}M = E_i \oplus F_i$ is not dominated then, after a series of perturbations, we obtain a new linear system having non-real eigenvalues of every rank $(i, i+1)$, $i \in \{1, \ldots, n-1\}$. Using this fact, the existence of the transitions allows us to intermingle the expansion/contraction in all the directions E_i: by some rotation (associated to a homothety) one send the E_i direction to the E_{i+1} direction (and vice-versa). Thus given any i and k one can map the E_i direction to the E_k direction. This allows to distribute the expansion/contraction of the system along all the E_i directions uniformly. This is the key for getting a point whose associated matrix is a homothety. For details, see [9, Section 5].

In fact, the previous result is a consequence of [9, Proposition 2.1], which states a dichotomy for (abstract) periodic linear systems with transitions: either the system $(\Sigma, f, \mathcal{A})$ has a dominated splitting or there is a perturbation $\tilde{A}$ and a point x in the basis Σ such that $M_{\tilde{A}}(x)$ is a homothety. In Section 4.3, we will discuss the proof of this result in the two dimensional case.

The homothety we obtained above corresponds to a linear system which is the perturbation of a *dynamical* linear system. A priori, such a homothety has no dynamical meaning. Next key lemma allows us to interpret dynamically this homothety:

Lemma 4.1 (Franks, [23]). *Suppose that Γ is an $f-$invariant finite set and B is an ε-perturbation of the derivative of f along the set Γ. Then for every $\varepsilon > 0$ and every neighbourhood U of Γ there is a diffeomorphism g ε-$\mathcal{C}^1$-close to f such that*

- *$f(x) = g(x)$ for all $x \in \Gamma$ or $x \notin U$,*
- *the derivative of $Dg(x) = B(x)$, for all $x \in \Gamma$.*

This lemma allows us to consider perturbations of the derivative of f keeping unchanged the dynamics of f over a periodic orbit. In this way, after a perturbation, one gets a suitable derivative along some periodic orbits.

The interesting dynamical consequence of the previous construction (lack of domination generates homotheties) and Lemma 4.1 is that if a homoclinic class does not admit any dominated splitting then one can get a homothety after a perturbation of the derivative along some orbit of the class. The key result is now the following:

Lemma 4.2 (Scattering Property, Lemmas 1.9 and 1.10 in [9]). *Consider a homoclinic class $H(P,f)$. Fix $\varepsilon > \varepsilon_0 > 0$ and assume that there are $x \in \Sigma$ and an ε_0 perturbation $\tilde{A}$ of the derivative of $Df^{\pi(x)}(x)$ such that $M_{\tilde{A}}(x)$ is either a dilatation (i.e. all its eigenvalues have modulus bigger than 1) or a contraction (i.e. all its eigenvalues have modulus less than 1).*

Then there are a point $y \in \Sigma_P$ and an ε-C^1-perturbation g of f such that y is a sink (resp. source) of g.

4.2 Non-dominated homoclinic classes and the coexistence phenomenon

As in the results in Section 2.2, the key step is to see that if the homoclinic class $H(P,f)$ does not admit a dominated splitting then there is a diffeomorphism g arbitrarily C^1-close to f with a sink or a source arbitrarily close to the homoclinic class $H(P_g,g)$, see [9, Theorem 1]. Assuming that the homoclinic class of P is persistently non-dominated, one gets a condition similar to the one of robust tangencies in the Newhouse's coexistence phenomenon. The genericity argument in this case follows similarly.

More precisely, consider an open set $\mathcal{U}$ such that the map $f \mapsto P_f$ which associates to a diffeomorphism $f \in \mathcal{U}$ a saddle P_f of f is continuous. Let $\mathcal{D}$ be the set of diffeomorphisms $g \in \mathcal{U}$ such that $H(P_g,g)$ has some dominated splitting. Denote by $\mathcal{C}$ the closure of the interior of $\mathcal{D}$. We now consider the set $\mathcal{H}$ defined as the interior of $\mathcal{C}$. Let $\mathcal{S}$ be the complement of $\mathcal{C}$. The union of the sets $\mathcal{H}$ and $\mathcal{S}$ is dense in $\mathcal{U}$. Clearly, the diffeomorphisms g such that the homoclinic class of P_g has a dominated splitting form a dense and open subset of $\mathcal{H}$. We claim that there is a residual subset of the open set $\mathcal{S}$ of diffeomorphisms having infinitely many sinks/sources. By Lemma 4.2, given any g in $\mathcal{S}$ there is h arbitrarily close having at least one sink/source (this is an open property). Thus the subset $\mathcal{S}_1$ of $\mathcal{S}$ of diffeomorphisms having at least one sink/source is open and dense in $\mathcal{S}$. It is now enough to define inductively the sets $\mathcal{S}_k$ (which are open and dense in $\mathcal{S}$) and consider the residual subset $\mathcal{S}_\infty = \cap \mathcal{S}_i$ of $\mathcal{S}$.

4.3 Two-dimensional linear systems

We now explain why the absence of domination generates (after a perturbation) non-real eigenvalues. We see this property in the simplest two dimensional case (in some sense, all arguments involved in the general case have a two dimensional flavor). In what follows, (Σ, f, A) is a periodic linear system, where $A(x) \in GL_+(2,\mathbb{R})$ for every $x \in \Sigma$. There are two reasons for an invariant splitting $E \oplus F$ fails of being dominated. First, the angle between its bundles may not be uniformly lower bounded. In this case, the following lemma (whose proof is straightforward) gives the non-real eigenvalues:

Lemma 4.3 (Lemma 3.2 in [9]). *For every $\alpha > 0$ and every matrix $M \in GL_+(2,\mathbb{R})$ with two different eigenspaces E_1 and E_2 whose angle is less than α there is $s \in [-1,1]$ such that $R_{s\,\alpha} \circ M$ has a pair of non-real eigenvalues (here $R_{t\,\alpha}$ denotes the rotation of angle $t\,\alpha$).*

The second reason concerns the expansion in the E and F directions. Assume that the angle between these bundles is uniformly bounded from below. Thus, after a bounded change of the metric, we can assume that these bundles are orthogonal. That is, $E = \mathbb{R} \times \{0\}$ and $F = \{0\} \times \mathbb{R}$. Thus the system consists of diagonal matrices. Given x in the basis Σ, denote by $\sigma(x)$ and $\lambda(x)$ the eigenvalues of $M_{[A]}(x)$ associated to the vertical direction $\{0\} \times \mathbb{R}$ and the horizontal direction $\mathbb{R} \times \{0\}$, respectively. Assume for simplicity that, for any $x \in \Sigma$, $|\sigma(x)| \geq |\lambda(x)|$.

Lemma 4.4 (Lemma 3.4 in [9]). *For any $\varepsilon > 0$ and $\alpha > 0$, there is $\ell \in \mathbb{N}$ with the following property. Suppose that the splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is not ℓ-dominated. Then there are an $\varepsilon-$perturbation $\tilde{A}$ of the matrices A and a point x of the basis such that the angle between the eigenspaces of $M_{\tilde{A}}(x)$ is less than α.*

Applying now Lemma 4.3, we get the announced non-real eigenvalues.

The proof of Lemma 4.4 goes as follows. Write

$$I_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

Remark 4.5 (Lemma 3.3 in [9]). *For every $\alpha > 0$ and $\mu > 0$, there is $c > 1$ such that for every pair of matrices*

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \quad \textit{and} \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad \textit{with} \quad \frac{|b_1|}{|b_2|} > c \quad \textit{and} \quad \frac{|b_1\, c_1|}{|b_2\, c_2|} < 1,$$

the angle between the eigenvectors of $D = B \circ I_\mu \circ C$ is less than α,

The heuristic idea of the proof of this remark is the following. Note that $(1, 0)$ is an eigenvector of D. Consider the vector $(1, \beta)$, for some *small* $0 < \beta \leq 2/(c\,\mu)$ fixed. As $|b_1/b_2|$ and $|c_2/c_1|$ are large (i.e., greater than c), the vectors $B^{-1}(1, \beta)$ and $C(1, \beta)$ are almost vertical (angle with the vertical less than μ). The matrix I_μ now sends the direction of $C(1, \beta)$ into the direction of $B^{-1}(1, \beta)$. Thus $(1, \beta)$ is an eigenvector of D.

To prove Lemma 4.4 write

$$A(x) = \begin{pmatrix} a(x) & 0 \\ 0 & b(x) \end{pmatrix}.$$

Suppose first that there is x in Σ such that there is no domination in the period:

$$|\sigma(x)| \leq (1 + \mu)^{2\,\pi(x)}\, |\lambda(x)|, \quad \text{where } \pi(x) \text{ is the period of } x.$$

Multiplying the matrices $A(f^i(x))$ by matrices of the form

$$\begin{pmatrix} 1 + \nu & 0 \\ 0 & \frac{1}{1+\nu} \end{pmatrix}, \quad \text{for some } \nu \in [0, \mu],$$

one gets a perturbation B of A such that $M_B(x)$ is a homothety. Thus given any pair of (different) directions of $\mathbb{R}^2$, there is a perturbation C of B such that the eigenvectors of $M_C(x)$ are parallel to such directions. This ends the proof of the lemma in this case.

Thus we can now assume that there is domination in the period:

$$|\sigma(x)| > (1 + \mu)^{2\,\pi(x)}\, |\lambda(x)|, \qquad \text{for every } x \in \Sigma.$$

Consider the constant c in Remark 4.5 (associated to α, μ, and ℓ) such that $(1+\mu)^\ell > 2\,c$.

Since the splitting given by the coordinate directions is not ℓ-dominated, there is x such that

$$2\,|\prod_0^{\ell-1} a(f^i(x))| \geq |\prod_0^{\ell-1} b(f^i(x))|.$$

Assume first that $\ell < \pi(x)$. Consider the perturbation $\tilde{A}$ of A given by

$$\tilde{A}(y) = \begin{pmatrix} \tilde{a}(y) & 0 \\ 0 & b(y) \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} \tilde{a}(y) = (1+\mu)\,a(y), \text{ if } y = f^i(x), i \in \{0, \cdots, \ell-1\}, \\ \tilde{a}(y) = a(y), \text{ if } i \in \{\ell, \dots, \pi(x)-1\}. \end{array}$$

Let

$$B = \prod_0^{\ell-1} \tilde{A}(f^i(x)) \quad \text{and} \quad C = \prod_\ell^{\pi(x)-1} \tilde{A}(f^i(x)).$$

Then we have

$$M_{\tilde{A}}(x) = C \circ B \quad \text{and} \quad M_{\tilde{A}}(f^\ell(x)) = B \circ C.$$

Observe that B and C verify the hypotheses in Remark 4.5. Thus the angle between the eigenvectors of $D = B \circ I_\mu \circ C$ is less than α. This gives the announced perturbation.

It remains to consider the case $\ell \geq \pi(x)$. Since ℓ cannot be a multiple of $\pi(x)$ one has $\ell = k\,\pi(x) + l_0$, for some $k \geq 1$ and $1 \leq \ell_0 < \pi(x)$. By hypothesis (domination in the period),

$$(1+\mu)^{2\,\pi(x)}\,|\lambda(x)| = (1+\mu)^{2\,\pi(x)}\,|\prod_0^{\pi(x)-1} a(f^i(x))| < |\prod_0^{\pi(x)-1} b(f^i(x))| = |\sigma(x)|.$$

Moreover, since the splitting is not ℓ-dominated, we also have

$$2\,|\prod_0^{\ell-1} a(f^i(x))| \geq |\prod_0^{\ell-1} b(f^i(x))|.$$

Thus

$$|\prod_0^{\ell_0-1} a(f^i(x))| > \frac{(1+\mu)^{2\,k\,\pi(x)}}{2}\,|\prod_0^{\ell_0-1} b(f^i(x))|.$$

Finally, note that $2\,k\,\pi(x) > \ell$ and recall that $(1+\mu)^\ell > 2\,c$, thus

$$\frac{(1+\mu)^{2\,k\,\pi(x)}}{2} > \frac{(1+\mu)^l}{2} > c.$$

This implies that, as in the previous case, we can apply Remark 4.5 to

$$B = \prod_0^{\ell_0-1} \tilde{A}(f^i(x)) \quad \text{and} \quad C = \prod_{\ell_0}^{\pi(x)} \tilde{A}(f^i(x)).$$

This gives a perturbation such that the angles of the eigenspaces is less than α. The proof now follows from Lemma 4.3.

5 Strong forms of the coexistence phenomenon

In this section, we consider non-dominated dynamical configurations (as the ones in Section 4), under some extra mild hypotheses, we will derive some new coexistence features from it.

Suppose that $\mathcal{U}$ is an open set of C^1-diffeomorphisms such that there is a continuous map $\mathfrak{P}\colon \mathcal{U} \to M$ which associates to each f a saddle $\mathfrak{P}(f) = P_f$ whose homoclinic class does not admit any dominated splitting. Theorem 5 implies that there is a residual subset $\mathcal{R}_\mathcal{U}$ of $\mathcal{U}$ consisting of diffeomorphisms having infinitely many sinks or sources. In fact, if the Jacobian of $Df^{\pi(P_f)}P_f$ is bigger than one (for every $f \in \mathcal{U}$) then the set $\mathcal{R}_\mathcal{U}$ can be taken consisting of diffeomorphisms with infinitely many sources. Note that if, simultaneously, there is some saddle Q_f homoclinically related to P_f (for all $f \in \mathcal{U}$) such that the Jacobian of $Df^{\pi(Q_f)}Q_f$ is less than one (for all $f \in \mathcal{U}$) then there is there a residual subset $\mathcal{S}_\mathcal{U}$ of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks. Thus the diffeomorphisms in the residual subset $\mathcal{K}_\mathcal{U} = \mathcal{S}_\mathcal{U} \cap \mathcal{R}_\mathcal{U}$ of $\mathcal{U}$ have simultaneously infinitely many sinks and infinitely many sources. In fact, in this case the set $\mathcal{K}_\mathcal{U}$ can be taken satisfying a much stronger form of the coexistence phenomenon we proceed to describe.

We say that a homoclinic class $H(P_f, f)$ is *wild* if there is a C^1-neighbourhood $\mathcal{U}$ of f such that, for every g in $\mathcal{U}$, the continuation P_g of P_f is defined and its homoclinic class $H(P_g, g)$ satisfies the following two conditions:

W1) it contains a pair of saddles homoclinically related to P_g whose Jacobians are greater and less than one;

W2) it does not admit any dominated splitting.

Theorem 6 (Strong forms of the coexistence phenomenon, [8]). *Let $\mathcal{W}$ be an open set of (three dimensional) C^1-diffeomorphisms f having a wild homoclinic class $H(P_f, f)$. Then there is a residual subset $\mathcal{R}$ of $\mathcal{W}$ such that, for every $g \in \mathcal{R}$, the set $H(P_g, g)$ is simultaneously contained in the closure of infinitely many pairwise disjoint:*

- *saturated transitive sets with minimal dynamics,*
- *non-trivial uniformly hyperbolic attractors and repellers,*
- *non-trivial partially hyperbolic attractors and repellers,*
- *wild homoclinic classes,*
- *infinitely many sinks and sources.*

Let us recall that an (infinite) f-invariant closed set Λ is *minimal* if every orbit of it is dense in the whole Λ (in particular, Λ does not contain periodic points). We say that a transitive set Υ is *saturated* if it contains any transitive set intersecting it (thus these sets are maximal transitive, for the discussion of these notions see [11, Chapter 10]).

The proof of Theorem 6 follows using some of the arguments in Section 4. In the proof of Theorem 5, if there is some saddle in the non-dominated homoclinic class $H(P_f, f)$ (homoclinically related to P_f) whose Jacobian is less than one then the homothety can be chosen contracting. Thus under the hypotheses of Theorem 6, we can obtain simultaneously a contracting and an expanding homothety. Moreover, the existence of transitions allows us to intermingle the actions of these homotheties. In this way, we obtain a perturbation of the derivative at some saddle of the class which is exactly the identity. This property leads to the following dynamical property:

Theorem 7 (Universal Dynamics, [8]). *C^1-generic diffeomorphisms f with a wild homoclinic class satisfy the following universal property: for every open set $\mathcal{O}$ of diffeomorphisms of the disk $\mathbb{D}^3$, there are infinitely many disjoint periodic disks on which the first return map of f is conjugate to some element of $\mathcal{O}$.*

A consequence of the universal dynamics property is the following principle, see [8]:

Every robust property of C^1-diffeomorphisms of the disk D^3 is displayed infinitely many times in periodic disks of disjoint orbits for these locally generic diffeomorphisms having a wild homoclinic class. In particular, they exhibit simultaneously infinitely many pairwise disjoint non-trivial homoclinic classes, infinitely many non-trivial hyperbolic and non-hyperbolic attractors, and infinitely many non-trivial repellors.

Denote by $\mathcal{W}(M)$ the set of diffeomorphisms of $\text{Diff}^1(M)$ having a wild homoclinic class. Then the set $\mathcal{U}(M)$ of diffeomorphisms with universal dynamics is residual in this set, [8, Theorem B]. Now to get the minimal sets in Theorem 6 one proceeds as follows. Given $f \in \mathcal{U}(M)$, there are natural numbers n_k, $n_k \to \infty$, and of nested disks D_k such that

- the disks $D_k, f(D_k), \dots, f^{n_k-1}(D_k)$ are pairwise disjoint,
- the diameters of the disks $f^i(D_k)$, $i = 1, \dots, n_k$, go to zero as $k \to \infty$,
- $f^{n_k}(D_k)$ is contained in the interior of D_k.

This construction gives an *infinitely renormalizable* invariant set

$$\Phi(f) = \bigcap_k \Delta_k, \quad \text{where} \quad \Delta_k = \bigcup_{i=0}^{n_k-1} f^i(D_k).$$

By construction, the Cantor set $\Phi(f)$ is minimal and maximal transitive. This construction is depicted in the Figure 10.

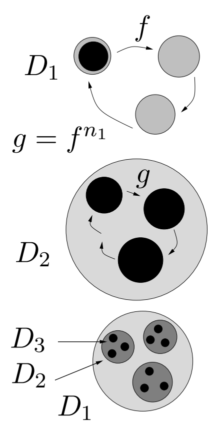

Figure 10: Infinitely renormalizable minimal sets

In the next section, we construct diffeomorphisms with robustly non-dominated homoclinic classes, obtaining in this way C^1-open sets of diffeomorphisms where the coexistence phenomenon occurs.

6 Generation of non-dominated homoclinic classes: heterodimensional tangencies

In view of the results in Section 4 and 5, a key problem is to construct an open set of diffeomorphisms $\mathcal{U}$ having a saddle P_f (depending continuously on the dynamics) whose homoclinic class does not admit any dominated splitting (this problem is open for C^1-surface diffeomorphisms). In Section 3, we introduced a dynamical configuration with robustly non-dominated homoclinic classes. In this section, we describe a much more natural method for obtaining non-dominated dynamics: heterodimensional tangencies.

The dynamical configuration of heterodimensional tangencies is reminiscent of the one of homoclinic tangencies. Instead of bifurcations of hyperbolic homoclinic classes as in Section 2.2, we consider partially hyperbolic ones. Typically, these non-hyperbolic classes contain saddles of indices one and two. We analyze what happens at heterodimensional tangencies relating saddles with different indices. This bifurcation provides a transition from *partially hyperbolic homoclinic classes to robustly non-dominated homoclinic classes.*

We consider three dimensional diffeomorphisms f with two saddles of P_f and Q_f such that $\operatorname{index}(P_f) = 1$, $\operatorname{index}(Q_f) = 2$, and $P_f \in H(Q_f, f)$. We say that f has a *heterodimensional tangency* associated to P_f and Q_f if $W^s(P_f, f)$ and $W^u(Q_f, f)$ have some non-transverse intersection at some point y_f. Note that $\dim(W^s(P_f, f)) + \dim(W^u(Q_f, f)) = 4$. We assume that the non-transverse intersection is parabolic: there are compact two-disks $K^s_f \subset W^s(P_f, f)$ and $K^u_f \subset W^u(Q_f, f)$ containing y_f in their interiors and local coordinates around $y_f = (0, 0, 0)$ such that, in these coordinates,

$$K^s_f = \{z = 0\} \quad \text{and} \quad K^u_f = \{z = x^2 + y^2\}.$$

For g close to f, we consider continuations $K^s_g \subset W^s(P_g, g)$ and $K^u_g \subset W^u(Q_g, g)$ of K^s_f and K^u_f. Clearly, there are diffeomorphisms g arbitrarily close to f such that $\sigma_g = K^s_g \cap K^u_g$ is a transverse intersection diffeomorphic to a circle.

We need a result about homoclinic classes of C^1-generic diffeomorphisms:

Proposition 6.1 (Lemma 2.1 in [2] following [4, 13]). *Consider a diffeomorphism f and any pair of (hyperbolic) saddles P_f and Q_f of f of different indices. Then there are a C^1-open neighbourhood $\mathcal{U}_f$ of f and a residual subset $\mathcal{R}_{\mathcal{U}_f}$ of $\mathcal{U}_f$ with the following property:*

- *either $H(P_g, g) = H(Q_g, g)$, for all $g \in \mathcal{R}_{\mathcal{U}_f}$;*
- *or $H(P_g, g) \cap H(Q_g, g) = \emptyset$, for all $g \in \mathcal{R}_{\mathcal{U}_f}$.*

In the first case, we say that the saddles P_f and Q_f are persistently linked (in $\mathcal{U}_f$).

This proposition, in particular, implies that the open set $\mathcal{C}$ in Section 3 contains a residual subset $\mathcal{R}$ such that $H(P_f, f) = H(Q_f, f)$, for all $f \in \mathcal{R}$.

Theorem 8 ([16]). *Let $\mathcal{L}_{P,Q}(M)$ be an open set of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{L}_{P,Q}(M)$, there are saddles P_f and Q_f (depending continuously on f) of indices one and two which are persistently linked in $\mathcal{L}_{P,Q}(M)$. Assume that there is $h \in \mathcal{L}_{P,Q}(M)$ with a heterodimensional tangency associated to P_h and Q_h. Then there is a C^1-open set $\mathcal{V}$ such that its closure contains h and the homoclinic class $H(Q_g, g)$ does not admit any dominated splitting for every $g \in \mathcal{V}$.*

The main step of the proof of this proposition is to check that, if $g \in \mathcal{V}$, the small circles $\sigma_g = K^s_g \cap K^u_g$ of the transverse intersection of $W^s(P_g, g)$ and $W^u(Q_g, g)$ are contained in the homoclinic class of Q_g. We will explain this property in the example in Section 6.1

The first step is to see that one can assume that the heterodimensional tangency is associated to saddles whose multipliers are all real and positive and have multiplicity one (this follows from the results in Section 4: generically, the sets $\mathrm{Per}_{\mathbb{R}}(P_g)$ and $\mathrm{Per}_{\mathbb{R}}(Q_g)$ are dense in the homoclinic classes of P_g and Q_g).

Consider now g such that σ_g is contained in the homoclinic class $H(Q_g, g)$ and assume, by contradiction, that the class has a dominated splitting $E \oplus F$. Suppose, for instance, that $\dim(E) = 1$. For each $x \in H(Q_g, g)$, consider the fibers E_x and F_x and the angle $\alpha(x)$ between E_x and F_x. The domination of $E \oplus F$ implies that $\alpha(x) > \alpha_0$, for all $x \in H(Q_g, g)$. Assume, for simplicity, that $g^k(\sigma_g) \to P_g$ and $g^{-k}(\sigma_g) \to Q_g$ as $k \to \infty$. Since $H(Q_g, g)$ is closed and g-invariant, one has that $P_g \in H(Q_g, g)$.

Consider $\Sigma_g = \cup_k g^k(\sigma)$ and the g-invariant family of bundles $\mathcal{H} = \{H(x)\}_{x \in \Sigma_g}$, $H_x = T_x \Sigma_g$. The domination of $E \oplus F$ and the invariance of the family $\mathcal{H}$ imply that either $H_x \subset E_x$ for all $x \in \Sigma_g$, or $H_x \subset F_x$ for all $x \in \Sigma_g$.

Since the curves $g^k(\sigma)$ are arbitrarily small circles close to P_g (in a small local stable manifold of P), for each big k there are points $z, w \in g^k(\sigma_g)$ such that: (a) H_z is *parallel* to $E^{ss}_P = E_P$ and (b) H_w is *parallel* to E^c_P, where E^c_P is the center-stable space of P. See Figure 11. Note that, since $\dim(E) = 1$, we have $E^c_P \subset F_P$.

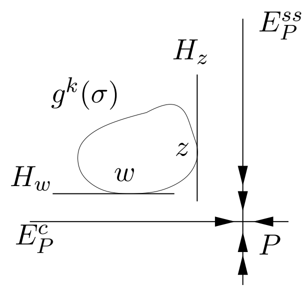

Figure 11: The splitting cannot be dominated

First, if $H_x \subset E_x$ for all $x \in \Sigma_g$ (thus $H_x = E_x$) then (b) above a contradiction: there is w close to P such that $E_P = E^{ss}_P$ and $E_w = H_w$ are not close. Finally, if $H_x \subset F_x$ then (a) gives a contradiction: there is z close to P such that the angle between F_z and E_z is close to zero, contradicting that $\alpha(z) > \alpha_0 > 0$.

6.1 Examples of diffeomorphisms with heterodimensional tangencies

In this section, we construct diffeomorphisms with heterodimensional tangencies obtained as *bifurcations* of partially hyperbolic sets. Our construction is motivated (and it is also similar) to the one in [35, page 89-90] for diffeomorphisms with homoclinic tangencies (see also the constructions in Section 2.2). Let us recall the main steps of the construction in [35].

Consider a diffeomorphism f having a Plykin attractor Σ in a disk D of $\mathbb{R}^2$. Take a fixed saddle P of f in Σ and note that attractor Σ is the homoclinic class of P. Multiply now f by a linear expansion $g(x) = \lambda\, x$ on the line $\mathbb{R}$. We take large λ in such a way $D \times \{0\}$ is normally hyperbolic (any expansion of the derivative of g in the disk D is strictly upper bounded by λ). Note that the point $S = (P, 0)$ is a saddle of index two of the product diffeomorphism $\phi = (f, g)$ and that $\Lambda = \Sigma \times \{0\} \subset D \times \{0\}$ is a hyperbolic transitive set of ϕ.

We pick points Z_1 in the strong unstable manifold of S (tangent to $\{0, 0\} \times \mathbb{R}$) and Z_2 in the stable manifold of S and perform a semi-local perturbation of ϕ preserving the hyperbolic set Λ: we consider an arc $(\phi_t)_{t \in [0,1]}$, with $\phi_0 = \phi$, such that for every $t \in [0, 1)$ the set Λ is a hyperbolic set of ϕ_t and, for $t = 1$, ϕ_1 has a homoclinic tangency associated to S at the point A_2. The perturbation is depicted in Figure 12. A key condition here is that Λ does not contain points in the region $z > 0$.

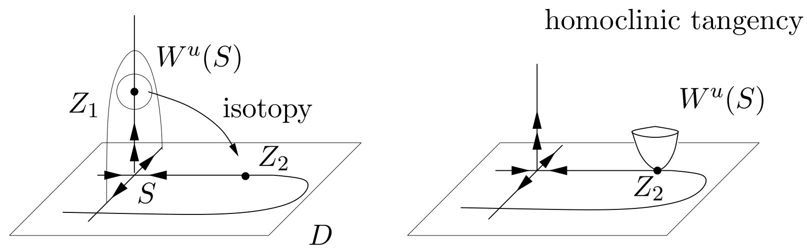

Figure 12: Creation of a homoclinic tangency

We now sketch our construction of diffeomorphisms with a heterodimensional tangency. For clearness, we use coordinates in $\mathbb{R}^3$, but our construction holds for much more general diffeomorphisms. We consider a diffeomorphism f with a heterodimensional cycle related to saddles $Q = (0, 0, 0)$ and $P = (0, 4, 0)$, of indices 2 and 1, such that dynamics around the cycle is the following. In the cube $R = [-1, 1] \times [-1, 5] \times [-1, 1]$,

$$f(x, y, z) = (\lambda_s\, x, F(y), \lambda_u\, z), \qquad 0 < \lambda_s < 1 < \lambda_u,$$

where F is a strictly increasing function such that $F'(0) = \beta > 1 > F'(4) = \lambda > 0$, and $\lambda_s < F'(x) < \lambda^u$, for all $x \in [-1, 5]$. Note that he restriction of f to the cube R has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with three non-trivial bundles (given by the coordinate directions). Moreover the stable manifold $W^s(P)$ of P and the unstable manifold $W^u(Q)$ of Q meet transversely along the normally hyperbolic curve $\{0\} \times (0, 4) \times \{0\}$.

We fix the points $A = (0, 4, -\frac{1}{2}) \in W^u(P)$ and $B = (-\frac{1}{2}, 0, 0) \in W^s(Q)$ and assume that there is $k_0 \in \mathbb{N}$ with $f^{k_0}(A) = B$ and $f^j(A) \notin R$ for all $0 < j < k_0$. Hence $W^u(P) \cap W^s(Q) \neq \emptyset$. Thus f has a heterodimensional cycle associated to the saddles P and Q of indices one and two, respectively. We also assume that $W^u(P)$ and $W^s(Q)$ meet *quasi-transversely* throughout the orbit of the heteroclinic point B:

$$T_B W^u(P) + T_B W^s(Q) = T_B W^u(P) \oplus T_B W^s(Q) = \mathbb{XZ}.$$

This heterodimensional cycle is depicted in Figure 13. We can perform this construction in such a way that the resulting diffeomorphism (and the arcs unfolding the associated cycle) satisfies Theorem 4.

We take small neighbourhoods V of the heteroclinic curve $\{0\} \times [0, 4] \times \{0\}$ and U_A of A such that V is contained in R, $U_A, \dots, f^{k_0}(U_A)$ are pairwise disjoint, $f^{-1}(U_A) \subset V$, $f^{k_0+1}(U_A) \subset V$,

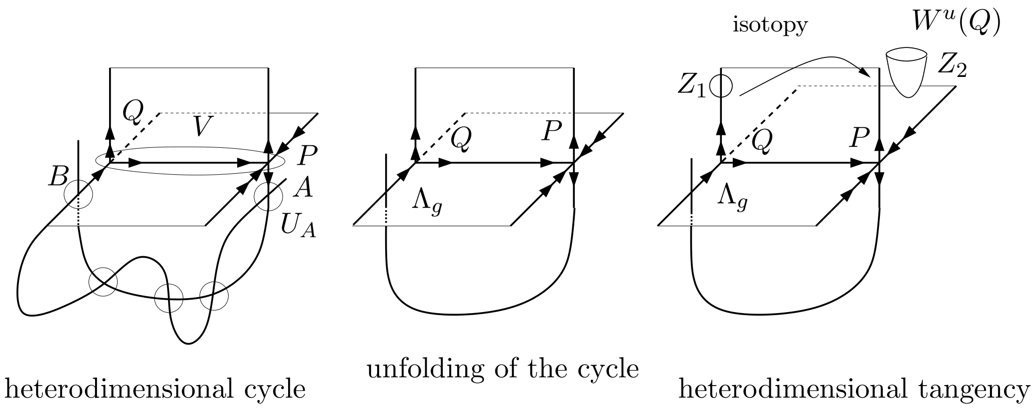

Figure 13: A heterodimensional tangency

and $f^j(U_A) \cap R = \emptyset$, for all $0 < j < k_0$. We select the neighbourhood of the cycle

$$W = V \cup U_A \cup f(U_A) \cup \cdots \cup f^{k_0}(U_A).$$

Given g close to f, let $\Lambda_g = \cap_{i\in\mathbb{Z}} g^i(W)$ be the maximal invariant set of g in W. By construction, the set Λ_g is partially hyperbolic. We also can take g such that there are transverse homoclinic points of P and Q close to A and B (i.e., this follows by unfolding the heterodimensional cycle). Moreover, this is much more relevant, the diffeomorphism g can be taken such the set Λ_g is a transitive set and equal to the homoclinic class of Q, recall Theorem 4 and the discussion in Section 3. In particular, this set contains the (continuation) of heteroclinic segment $\{0\}\times[0,4]\times\{0\}$. Thus $P \in \Lambda_g$. In fact, the set Λ_g is a prototype of *robustly transitive set*. Thus the set $\Lambda_g = H(Q,g)$ verifies the hypotheses of Theorem 8.

We now take h close to g such that $\Lambda_h = H(Q,h)$ is robustly transitive. We can now use the arguments above to get a heterodimensional tangency associated to P and Q. A key point here is that the set Λ_h does not intersect the regions $z > 0$ and $x < 0$. We now exactly repeat the arguments above taking, for instance, $Z_1 = (0,0,1/2) \in W^u(Q,h)$ and $Z_2 = (-1/2,1/2,0) \in W^s(P,h)$. As above, there is an arc $(h_t)_{t\in[0,1]}$, with $h_0 = h$, such that for every $t \in [0,1)$ the set $\Lambda_{h_t} = H(Q,h_t)$ is robustly transitive (we do not modify the dynamics around the cycle) and for $t = 1$, h_1 has a heterodimensional tangency associated to Q and P at the point Z_2. See Figure 13. This ends our construction.

We finally explain how the *unfolding of the heterodimensional tangency* generates C^1-robustly non-dominated homoclinic classes. The proof of the transitivity of Λ_g involves the distinctive property of the heterodimensional cycles in Section 3: consider any 2-disk Δ intersecting transversely the two-dimensional stable manifold of P along a curve, then the one-dimensional stable manifold of Q transversely intersects Δ. This implies that the curve $\{0\} \times [0,4] \times \{0\}$ is contained in the homoclinic class of Q: given any point $Z \in \{0\} \times [0,4] \times \{0\}$ there is an arbitrarily small two-disk $\Delta(Z)$ contained in the unstable manifold of Q and whose interior contains Z. Then, by the distinctive property, this disk transversely meets $W^s(Q)$, thus it contains a point of the homoclinic class of Q. Since the disk $\Delta(Z)$ can be chosen arbitrarily small and a homoclinic class is closed, we have $Z \in H(Q,g)$ for all $Z \in \{0\} \times [0,4] \times \{0\}$.

Unfolding the heterodimensional tangency of h_1, we obtain diffeomorphisms φ such that the transverse intersection between $W^s(P,\varphi)$ and $W^u(Q,\varphi)$ contains small circles σ. Repeating the argument before (i.e., applying the distinctive property) for points $Z \in \sigma$, we have that the circle σ is contained in the homoclinic class $H(Q,\varphi)$. This implies that the saddle P (resp. Q) is accumulated by arbitrarily small circles contained in the homoclinic class of Q. These circles are also contained in the intersection of the stable manifold of P and the unstable manifold of Q. As we showed above, these circles prevent the existence of a dominated splitting over the homoclinic class of Q.

Noting that previous arguments hold for diffeomorphisms φ close to h_1 such that the intersection between $W^s(P,\varphi)$ and $W^u(Q,\varphi)$ contains small circles, we get an open set $\mathcal{U}$ of diffeomorphism having a non-dominated homoclinic class.

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