“non-hyperbolicities”

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goals, questions

How to characterize the absence of (uniform) hyperbolicity?

What structures cannot exist in the hyperbolic case but must be present in its complement?

general facts

Many nonhyperbolic systems exhibit “some (weak) hyperbolicity.”

non-uniform, partial, singular, dominated splittings....

A little hyperbolicity goes a long way (Pugh-Shub).
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Palis conjecture

Dichotomy: hyperbolicity versus cycles

cycles:

- homoclinic tangencies (dim ≥ 2),
- heterodimensional cycles (dim ≥ 3).
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the conjecture holds for

- circle maps (Peixoto),
- $C^1$ surface diffeomorphisms (Pujals-Sambarino)
- $C^1$ tame diffeomorphisms $n \geq 3$ (Bonatti-D.).

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Those having stably finitely many homoclinic classes.

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Beyond hyperbolicity, non-hyperbolic features, $C^1$-generic setting, weak hyperbolicities, and non-hyperbolic features.

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### Tame Diffeomorphisms, $n \geq 3$

Dichotomy: hyperbolicity versus robust heterodimensional cycles.
Heterodimensional cycles associated to hyperbolic sets

Λ and Σ hyperbolic basic sets, different indices

\[ W^s(\Lambda) \cap W^u(\Sigma) \neq \emptyset \quad \text{and} \quad W^u(\Lambda) \cap W^s(\Sigma) \neq \emptyset. \]

similarly for homoclinic tangencies.

robust cycles (heterodim. cycles and tangencies)

every \( g \) close to \( f \) has a cycle.

Kupka-Smale Theorem

generically, periodic points are hyperbolic and their invariant manifolds are in general position (transversality).

thus: robust cycles involve some non-trivial hyperbolic set.
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some non-hyperbolic features
- bifurcations of periodic orbits (saddle-node, flip, Hopf),
- absence of shadowing properties,
- cycles,
- Newhouse-like phenomena: super-exponential growth of the number of periodic points,
- non-hyperbolic ergodic measures with large support,
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diffeo $f : M \to M$, $\dim M = n$,

$\mu$ ergodic measure of $f$: $\mu(f^{-1}(A)) = A$ implies $\mu(A) = 0, 1$,

there are $\Lambda$ of full $\mu$-measure,

$\chi_\mu^1 \leq \chi_\mu^2 \leq \cdots \leq \chi_\mu^n$

for all $x \in \Lambda$ and all $v \in T_x M$, $v \neq 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log ||Df^n(v)|| = \chi_\mu^i,$$

some $i = 1, \ldots, n$.

$\chi_\mu^i$ is the $i$-th Lyapunov exponent of $\mu$.

$\mu$ is non-hyperbolic if $\chi_\mu^i = 0$ for some $i$. 
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A-M maps are dense in the space of $C^r$-maps but no $C^r$-generic, $r \geq 1$ (Kaloshin).
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symbolic extensions

\[ f : X \rightarrow X, \text{ homeomorphism,} \]

\((X, f)\) has a **symbolic extension** if there exists a subshift (finitely many symbols) \((\Sigma, \sigma)\) and a continuous surjective map \(\pi : \Sigma \rightarrow X\) such that

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\((\Sigma, \sigma)\) is called an **extension** of \((X, f)\)

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$C^1$-diffeomorphisms,

context: $C^1$-generic dynamics (i.e., study of (locally) residual subsets)

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\( f : M \to M, \) diffeo., \( M \) compact and closed,

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dominated splitting: \( T_\Lambda M = E \oplus F, \) \( Df \)-invariant and there is \( m \) with

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\frac{|Df^m(v)|}{|(Df^m(w))|} \leq \frac{1}{2}
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for every unitary vectors \( v \in E_x \) and \( w \in F_x \) and all \( x \in \Lambda \).
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**caution**: a homoclinic class whose saddles have all the same index may be non-hyperbolic....
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for simplicity: **dimension of** \( M \) **is three**

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<tr>
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<td>• Robust het. cycles</td>
<td>Yes</td>
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<td>• non-hyperbolic measures</td>
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<td>- full support</td>
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<td>3?</td>
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