ANALYTIC CONTINUATION IS IMPRACTICAL

Nicolau Corção Saldanha

A DISSERTATION

PRESENTED TO THE FACULTY

OF PRINCETON UNIVERSITY

IN CANDIDACY FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE

BY THE DEPARTMENT OF

MATHEMATICS

June 1989

ABSTRACT

In this work we study the space $\mathcal{M}$ of analytic functions from the disc to the disc. For a metric space M we define a family of numbers $\Gamma_{n,r}(M)$ which give several ways of measuring the "size" of the metric space. $\Gamma_{1,1}$ is the Hausdorff dimension and we accordingly think of $\Gamma_{n,r}$ as a generalization of this concept. We call $\Gamma_{1,r}$ the generalized dimension and $\Gamma_{2,1}$ the hyperdimension of the space. We compute the hyperdimension and generalized dimension of $\mathcal{M}$ with several metrics; the simplest is $d_A(f,g) = \sup_{z\in A} |f(z) - g(z)|$ where A is a compact subset of the open unit disc. The hyperdimension of $\mathcal{M}$ turns out to be 2 for most "natural" metrics. The generalized dimension, however, is much more interesting and much harder to compute. If $\mathcal{M}$ is given the metric d_A defined above we have

$$\Gamma_{1,2}(\mathcal{M}) = \frac{1}{2\pi}\mathbf{C}(\Delta - A; \partial\Delta, \partial A)$$

where $\mathbf{C}$ denotes the conductivity of a surface in the sense given by electricity. We generalize d_A to d_p where p is a function assuming non-negative real values on the open disc; we give estimates of

$$\Gamma_{1,2}(\mathcal{M}, d_p).$$

After this part of the work was completed, we found out that Kolmogorov had intoduced the closely related concept of the entropy of a metric space and that Erohin had already proved a very similar version of the result concerning d_A.

We apply these results to the problem of specifying analytic functions. We prove certain inequalities that bound the efficiency of any specification procedure and exhibit one procedure that realizes the corresponding equality. As we found out about Kolmogorov's and Erohin's work we discovered that these ideas had already been pursued by Vituškin.

Our main application is the study of the problem of doing analytic continuation. We know that analytic continuation can always be done "in theory". We show that it is very hard to do it "in practice" in the following sense: the amount of information about the function which we need in order to perform the analytic continuation within an error of ϵ grows quadratically with $|\log(\epsilon)|$ but exponentially with the "length" of the arc along which analytic continuation is to be done. By the "length" of the arc we mean its length in the Kobayashi metric on the domain where the function is known to be defined. All of this shows that analytic continuation is impractical in all but the simplest cases; explicit examples are given.

Contents

Chapter 0

Introduction

The original motivation for this work was the problem of analytic continuation. We all know that analytic continuation is theoretically possible in the following sense: if the value of an analytic function is given on a small region the value of the function is determined everywhere. Finding this value, however, is a different story.

This motivates the following questions. The first question is: given the value of a function in a small region how can we compute the value of the same function outside that region? More than one answer is known for this question. All of these answers have the shortcoming, however, of being very inefficient in many situations. This brings us to other questions. Why are these methods inefficient? Is it possible that far better methods exist but we have not been clever enough to find them? Or is there some intrinsic limitation on the 'quality' of any possible such methods? If so, what are these limitations? These are the questions which we attempt to answer.

There is another class of questions which also served as inspiration for this work. Here we think of an analytic function as known but we want to specify what the function is. Alternatively, we can immagine that we have some function in mind and we want to tell somebody else, possibly a computer, what this function is. What is a good way of doing this? How much information do we need to give?

We soon saw that in order to answer any of these questions we needed first of all to study spaces of holomorphic functions. For several reasons, including concreteness and simplicity, we decided that the spaces we needed to study would be spaces of bounded holomorphic functions on some given domain. If this domain is simply connected the Riemann mapping theorem tells us that by composition with a fixed function we can suppose that this domain is the unit disc. This brings us to the space $\mathcal{M}$ of analytic functions from the disc to the disc. Several interesting metrics can be considered for $\mathcal{M}$.

So now we had metric spaces and we were interested in measuring their 'size'. Our first idea was to compute the Hausdorff dimension: this, however, turned out to be infinite. The natural thing to do now seemed to be to define some concept similar to Hausdorff dimension which would apply to these large metric spaces. This led us to introduce first the concept of *generalized dimension* and later that of *hyperdimension*. The similarity of these concepts led us to unify them in the symbol $\Gamma_{n,r}$, a very general way of measuring the 'size' of a metric space which, even though never used to its full power, helps to unify several proofs. Summing up, for a metric space M we define a family of numbers $\Gamma_{n,r}(M)$ which give several ways of measuring the 'size' of the metric space. $\Gamma_{1,1}$ is the Hausdorff dimension and we accordingly think of $\Gamma_{n,r}$ as a generalization of this concept. We call $\Gamma_{1,r}$ the generalized dimension and $\Gamma_{2,1}$ the hyperdimension of the space.

At this point we proceeded to compute the hyperdimension and generalized dimension of $\mathcal{M}$ with several metrics; the simplest is $d_A(f,g) = \sup_{z\in A} |f(z)-g(z)|$ where A is a compact subset of the open unit disc. The hyperdimension of $\mathcal{M}$ turns out to be 2 for this and most 'natural' metrics. The generalized dimension, however, is much more interesting and much harder to compute. If $\mathcal{M}$ is given the metric d_A defined above we have

$$\Gamma_{1,2}(\mathcal{M}, d_A) = \frac{1}{2\pi}\mathbf{C}(\Delta - A; \partial\Delta, \partial A)$$

where $\mathbf{C}$ denotes the conductivity of a surface in the sense given by electricity.

We generalize the metric d_A by introducing d_p where p is a function assuming non-negative real values in the open unit disc. d_A is the same as d_{χ_A} where χ_A is the characteristic function of A. In general, p should be thought of as telling us the 'weight' of each point in the definition of d_p. Seen by itself, d_p may seem to be an artificial concept but it turns out to be very important in many applications. d_p is a metric in many cases, but not always. We conjecture that

$$\Gamma_{1,2}(\mathcal{M}, d_p) = \frac{1}{\pi}\mathrm{E}(\phi_p)$$

where ϕ_p is the smallest superharmonic function which is everywhere greater or equal than p and E stands for the *energy* of a function. We do not prove this conjecture but we prove estimates towards it which are sufficient for the applications.

At this point we turn back to one of our motivating questions: we consider how most efficiently to specify holomorphic functions. Our results concerning the 'size' of spaces give us inequalities that bound the efficiency of any specification procedure. More than this is true, however. The proof of the generalized dimension results provides us with a very explicit specification procedure. The above mentioned inequalities show that no method can be much better than ours since our procedure realizes the corresponding equality.

This may be a good moment to remark that after practically all of this work was completed we found out that Kolmogorov had intoduced the concepts of the entropy and capacity of a metric space which correspond very closely to our $\Gamma_{n,r}$. We show how our concepts relate to Kolmogorov's but, being already used to our own, we do not adopt his notation consistently. A lot of work has been done estimating the entropy and capacity of several metric spaces. In particular, we found out that Erohin had already proved what we call the 'Conductivity Theorem' in the case of A connected and had in fact generalized this result in many directions which we did not consider (see [2, 3]). Notice also that Erohin's proof is essentially the same as ours, only in a somewhat different language. We also found out about the work of Vituškin (see [6]) who addresses the question of specifying functions in many different situations and derives many interesting results. We believe, however, that many other questions addressed in this work, especially in connection with analytic continuation, are actually new. We thank Curt McMullen for his crucial help in finding out about this related work.

We finally turn back to the original problem of the practicality of analytical continuation.

We show that, although theoretically possible, analytic continuation is impractical in all but the simplest situations. The precision with which the function needs to be known at the original point or region in order to give a decent approximation at the final point or region may be such as to render the problem physically impossible. All of these rather vague statements are defined precisely, often in more than one way. More explicitly, in one of these approaches we show that the amount of information about the function which we need in order to perform the analytic continuation within an error of ϵ grows quadratically with $|\log(\epsilon)|$ but exponentially with the "length" of the arc along which analytic continuation is to be done. By the "length" of the arc we mean its length in the Kobayashi metric on the domain where the function is known to be defined. We also give explicit examples. Not all questions are answered, however, and several interesting conjectures are left to be answered in the future.

I would like to thank all those who helped in this work, specially Bill Thurston, without whom this work would not have been born.

Chapter 1

Measuring Spaces of Functions

1.0 Introduction

In this chapter we define the basic concepts which we shall study and use later. Among the most important is the generalized dimension of a metric space, denoted by $\Gamma_{n,r}$, which is a way of measuring the "size" of the space. After this part of the work was completed we leaned about Kolmogorov's definitions of the entropy H_ϵ and the capacity C_ϵ of a metric space; we show how our concepts relate to Kolmogorov's but, being already used to our own, we do not adopt his notation consistently. We apply these concepts to spaces of holomorphic functions and define $\gamma_{n,r}$, which gives an indirect way of measuring subsets of the complex unit disc.

1.1 Measuring Metric Spaces

1.1.1 Packing and Covering Size

We will be interested in ways of measuring the "size" of a metric space. The concepts below are probably the most obvious in this sense.

Definition 1.1.1 *Let S be a metric space with distance d and take $\epsilon > 0$.*

(i) $A \subseteq S$ is called an ϵ-packing set *iff*

$$x, y \in A, x \neq y \quad \Longrightarrow \quad d(x, y) \geq \epsilon.$$

The ϵ-packing size *is defined as*

$$NP_\epsilon(S, d) = \max_{A \subseteq S \text{ an } \epsilon\text{-packing set}} |A|.$$

(ii) $B \subseteq S$ is called an ϵ-covering set *iff*

$$\forall x \in S \quad \exists y \in B \qquad d(x, y) \leq \epsilon.$$

The ϵ-covering size *is defined as*

$$NC_\epsilon(S, d) = \min_{B \subseteq S \text{ an } \epsilon\text{-covering set}} |B|.$$

The following proposition will be important later on:

Proposition 1.1.2 *If S is a metric space with distance d, then*

$$NC_\epsilon(S, d) \leq NP_\epsilon(S, d) \leq NC_{k\epsilon}(S, d)$$

for any $k < 1/2$. We will often take $k = 1/3$.

Proof of Proposition 1.1.2:

Let A be an ϵ-packing set of maximal size. We claim that A is an ϵ-covering set. In order to see this, pick any x in S. If $x \in A$, there obviously exists an element of A which is at a distance of at most ϵ from x, namely x itself. Otherwise, by maximality, we know that $A \cup \{x\}$ is *not* an ϵ-packing set. This means that there exist $y, z \in A \cup \{x\}$, $y \neq z$, with $d(y, z) < \epsilon$. Now we can not have $y, z \in A$, since A *is* an ϵ-packing set. We can therefore suppose without loss of generality $x = z$, and this shows that there exists an element of A which is at a distance of at most ϵ from x, namely y. This takes care of the first inequality.

Now let A be an ϵ-packing set and let B be an $k\epsilon$-covering set. Let us build a function $f : A \to B$ which takes each $x \in A$ to some $y \in B$ such that $d(x, y) \leq k\epsilon$; such an f exists since B is an $k\epsilon$-covering set. f is injective since $f(x_0) = f(x_1)$ implies $d(x_0, x_1) \leq d(x_0, f(x_0)) + d(x_1, f(x_1)) \leq 2k\epsilon < \epsilon$, which implies $x_0 = x_1$ by the fact that A is an ϵ-packing set. This takes care of the second inequality.

$\boxed{1.1.2}$

1.1.2 Hyperdimension and Generalized Dimension

We know that if S is nice enough we have

$$\lim_{\epsilon \to 0} \frac{\log NP_\epsilon}{|\log \epsilon|} = \lim_{\epsilon \to 0} \frac{\log NC_\epsilon}{|\log \epsilon|} = h(S),$$

where $h(S)$ is the Hausdorff dimension of S. The Hausdorff dimension is therefore a way of measuring the "size" of a metric space.

In this work, however, we will often be dealing with spaces with infinite Hausdorff dimension. This means that comparing $\log NP_\epsilon$ or $\log NC_\epsilon$ with $|\log \epsilon|$ is not a good idea. Kolmogorov's approach here is to define $H_\epsilon = \log NC_\epsilon$ and $C_\epsilon = \log NP_\epsilon$ and then to find estimates for these two functions. H is called the *entropy* and C the *capacity* of the metric space. Being used to our own definitions, we shall not use this terminology in most of this work but we shall translate our main results into this language.

Our approach is slightly different. We say that we should be comparing $\log NP_\epsilon$ or $\log NC_\epsilon$ with some function of ϵ that grows faster than $|\log \epsilon|$. One more or less natural candidate is $|\log \epsilon|^r$, where r is a real parameter; it will later become clear that this is the "right" function for many of our examples. The first question would be what value of r to take; given a value of r we can define an analogue of the Hausdorff dimension. Following these ideas and generalizing them we shall now define $\Gamma_{n,r}$, something much more general than we actually need. Before doing so, however, let us define (for local use) $\log^{\{n\}}(x) = \log(\log(\ldots(\log(x))\ldots))$ where we have n 'log' in the right hand side; more formally, $\log^{\{0\}}(x) = x$ and $\log^{\{n+1\}}(x) = \log(\log^{\{n\}}(x))$. Notice that $\log^{\{n\}}(x)$ is defined for large enough x and goes to infinity as x goes to infinity.

Definition 1.1.3 *Let S be a metric space with distance d. Let $n \geq 1$ be an integer and r, $0 < r < +\infty$, be a real number.*

We define:

$$\Gamma^+_{n,r}(S,d) = \limsup_{\epsilon\to 0} \frac{\log^{\{n\}} NP_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r} = \limsup_{\epsilon\to 0} \frac{\log^{\{n\}} NC_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r}$$

$$\Gamma^-_{n,r}(S,d) = \liminf_{\epsilon\to 0} \frac{\log^{\{n\}} NP_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r} = \liminf_{\epsilon\to 0} \frac{\log^{\{n\}} NC_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r}$$

When $\Gamma^+_{n,r}(S,d) = \Gamma^-_{n,r}(S,d)$ we denote this value by $\Gamma_{n,r}(S,d)$.

In particular, $\Gamma_{1,1}$ is the Hausdorff dimension. We call attention to the following special cases of this definition.

$$\Gamma^+_{2,1}(S,d) = \limsup_{\epsilon\to 0} \frac{\log\log NP_\epsilon(S,d)}{\log|\log\epsilon|} = \limsup_{\epsilon\to 0} \frac{\log\log NC_\epsilon(S,d)}{\log|\log\epsilon|}$$

$$\Gamma^-_{2,1}(S,d) = \liminf_{\epsilon\to 0} \frac{\log\log NP_\epsilon(S,d)}{\log|\log\epsilon|} = \liminf_{\epsilon\to 0} \frac{\log\log NC_\epsilon(S,d)}{\log|\log\epsilon|}$$

When $\Gamma^+_{2,1}(S,d) = \Gamma^-_{2,1}(S,d)$ we denote this value by $\Gamma_{2,1}(S,d)$. We call $\Gamma_{2,1}(S,d)$ the *hyperdimension* of S. This says (as we shall see) that $\Gamma_{2,1}(S,d)$ is the "most reasonable" value for r for which to consider $\Gamma_{1,r}(S,d)$.

$$\Gamma^+_{1,r}(S,d) = \limsup_{\epsilon\to 0} \frac{\log NP_\epsilon(S,d)}{|\log\epsilon|^r} = \limsup_{\epsilon\to 0} \frac{\log NC_\epsilon(S,d)}{|\log\epsilon|^r}$$

$$\Gamma^-_{1,r}(S,d) = \liminf_{\epsilon\to 0} \frac{\log NP_\epsilon(S,d)}{|\log\epsilon|^r} = \liminf_{\epsilon\to 0} \frac{\log NC_\epsilon(S,d)}{|\log\epsilon|^r}$$

When $\Gamma^+_{1,r}(S,d) = \Gamma^-_{1,r}(S,d)$ we denote this value by $\Gamma_{1,r}(S,d)$. We call $\Gamma_{1,r}(S,d)$ the *generalized dimension* of S The case $r = 2$ will be particularly important.

The next proposition tells us that this definition is legal.

Proposition 1.1.4 *Let S be a metric space with distance d. Then for any $n \geq 1$, $0 < r < +\infty$*

$$\limsup_{\epsilon\to 0} \frac{\log^{\{n\}} NP_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r} = \limsup_{\epsilon\to 0} \frac{\log^{\{n\}} NC_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r}$$

and

$$\liminf_{\epsilon\to 0} \frac{\log^{\{n\}} NP_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r} = \liminf_{\epsilon\to 0} \frac{\log^{\{n\}} NC_\epsilon(S,d)}{(\log^{\{n\}}(\epsilon^{-1}))^r}$$

Proof of Proposition 1.1.4:

From Proposition 1.1.2 we have $NP_\epsilon \geq NC_\epsilon$ from which we have the '$\geq$' inequality corresponding to each of the equations above.

But Proposition 1.1.2 also gives us $NP_\epsilon \leq NC_{\epsilon/3}$. This, together with

$$\lim_{\epsilon\to 0} \frac{(\log^{\{n\}}(3\epsilon^{-1}))^r}{(\log^{\{n\}}(\epsilon^{-1}))^r} = 1$$

gives us the '$\leq$' inequalities, concluding the proof of the proposition.

$\boxed{1.1.4}$

From now on we shall restrict our attention almost entirely to the cases $n = 1$ and $n = 2$, $r = 1$. Let us now prove a small result relating $\Gamma_{2,1}$ and $\Gamma_{1,r}$ corresponding to the intuitive observations made above.

Proposition 1.1.5 *Let (S,d) be a metric space. Then:*

$$\Gamma^-_{2,1}(S) > r \quad\Longrightarrow\quad \Gamma^-_{1,r}(S) = +\infty;$$

$$\Gamma^+_{2,1}(S) > r \quad\Longrightarrow\quad \Gamma^+_{1,r}(S) = +\infty;$$

$$\Gamma^-_{2,1}(S) < r \quad \Longrightarrow \quad \Gamma^-_{1,r}(S) = 0;$$

$$\Gamma^+_{2,1}(S) < r \quad \Longrightarrow \quad \Gamma^+_{1,r}(S) = 0.$$

Proof of Proposition 1.1.5:

This is an easy computation.

$\boxed{1.1.5}$

More generally, we would have

$$\Gamma^-_{n+1,1}(S) > r \quad \Longrightarrow \quad \Gamma^-_{n,r}(S) = +\infty$$

and similarly for the other cases but we shall refrain from needless generality.

Intuitively, $\Gamma_{1,r}$ is a variation of Hausdorff dimension designed for larger spaces and the "normal" situation should be the one where it is defined (i.e., where $\Gamma^+_{1,r} = \Gamma^-_{1,r}$). We are going to see that this indeed happens in the cases we are primarily interested in, but not always. $\Gamma_{2,1}$ is indeed some kind of a "hyperdimension", since it tells us what kind of generalized dimension is interesting for a given space. Going further, $\Gamma_{2,r}$ would be a generalized hyperdimension, $\Gamma_{3,r}$ would be a hyperhyperdimension and so on.

We can roughly understand the geometric meaning of $\Gamma_{n,r}$ in general and $\Gamma_{2,1}$ and $\Gamma_{1,r}$ in particular. If you look at your space with a microscope of resolution ϵ, you tend to think it has finite Hausdorff dimension and you make a guess as to what the dimension is. If you get a better microscope, however, you notice your first guess of the Hausdorff dimension was mistaken, and you make a higher guess. If you keep changing microscopes, your guesses are going to become larger and larger, probably tending towards infinity. $\Gamma_{2,1}$ and $\Gamma_{1,r}$ both measure how fast the guesses grow.

One important thing to remember about $\Gamma_{n,r}$ is that it isn't really very sensitive; the larger n is and, to a lesser extent, the larger r is, the less sensitive $\Gamma_{n,r}$ will be. Remember that the Hausdorff dimension was already incapable of telling a tiny disc from a huge one. For $r > 1$, $\Gamma_{1,r}$ is even less precise, since (very roughly speaking) it only measures the rate at which the Hausdorff dimension goes to infinity. A constant difference in the Hausdorff dimension goes undetected.

1.1.3 Equivalent Metrics

Sometimes a space S comes with more than one interesting metric. If the metrics are not too different, we can expect the two values of $\Gamma_{n,r}$ corresponding to the two metrics to be approximately equal. The next proposition formalizes this in one situation.

Definition 1.1.6 *If S is a space with two metrics d_0 and d_1 they are said to be* equivalent *if they satisfy*

$$C_0 d_0(x,y) \leq d_1(x,y) \leq C_1 d_0(x,y)$$

where $0 < C_0 \leq C_1 < +\infty$ are constants.

Proposition 1.1.7 *If S is a space with two equivalent metrics d_0 and d_1 then for any values of n and r*

$$\Gamma^-_{n,r}(S, d_0) = \Gamma^-_{n,r}(S, d_1), \qquad \Gamma^+_{n,r}(S, d_0) = \Gamma^+_{n,r}(S, d_1).$$

Proof of Proposition 1.1.7:

If A is an ϵ-packing set for d_0, it is a $(C_0\epsilon)$-packing set for d_1. It follows that $NP_\epsilon(S, d_0) \leq NP_{(C_0\epsilon)}(S, d_1)$. It is now easy to see (with only a few computations) that $\Gamma^-_{n,r}(S, d_0) \leq \Gamma^-_{n,r}(S, d_1)$ and $\Gamma^+_{n,r}(S, d_0) \leq \Gamma^+_{n,r}(S, d_1)$. The opposite inequalities are entirely analogous.

$\boxed{1.1.7}$

1.1.4 Product Spaces

Let us now prove another easy fact about Γ. Remember that if (S_0, d_0) and (S_1, d_1) are metric spaces there are at least three interesting metrics we can give to $S_0 \times S_1$, the maximum metric (L^∞), the sum metric (L^1) and the "square root of the sum of the squares" metric (L^2). Unless we say the opposite, the product of two vector spaces will receive the maximum metric. These metrics are all equivalent anyway, so if all we are interested in $\Gamma_{n,r}$, the choice hardly matters.

In order to see why the next proposition is the natural thing to expect, remember that $\Gamma_{1,r}$ is vaguely like a dimension, and that the dimension of the product of two spaces is the sum of the dimensions. Remember also that $\Gamma_{2,1}$ is some kind of a "hyperdimension".

Proposition 1.1.8 *If S_0 and S_1 are metric spaces and $S_0 \times S_1$ is their product,*

$$\max\Big(\Gamma^-_{2,1}(S_0); \Gamma^-_{2,1}(S_1)\Big) \leq \Gamma^-_{2,1}(S_0 \times S_1) \leq$$
$$\leq \Gamma^+_{2,1}(S_0 \times S_1) \leq \max\Big(\Gamma^+_{2,1}(S_0); \Gamma^+_{2,1}(S_1)\Big)$$

and, for any value of r,

$$\Gamma^-_{1,r}(S_0) + \Gamma^-_{1,r}(S_1) \leq \Gamma^-_{1,r}(S_0 \times S_1) \leq$$
$$\leq \Gamma^+_{1,r}(S_0 \times S_1) \leq \Gamma^+_{1,r}(S_0) + \Gamma^+_{1,r}(S_1).$$

When $\Gamma_{1,r}(S_0)$ and $\Gamma_{1,r}(S_1)$ are both defined, so is $\Gamma_{1,r}(S_0 \times S_1)$ and we have

$$\Gamma_{1,r}(S_0 \times S_1) = \Gamma_{1,r}(S_0) + \Gamma_{1,r}(S_1).$$

Also, if S is a metric space, then:

$$\Gamma^-_{2,1}(S^n) = \Gamma^-_{2,1}(S); \qquad \Gamma^+_{2,1}(S^n) = \Gamma^+_{2,1}(S);$$

$$\Gamma^-_{1,r}(S^n) = n\Gamma^-_{1,r}(S); \qquad \Gamma^+_{1,r}(S^n) = n\Gamma^+_{1,r}(S).$$

Notice that it is usually not true that

$$\Gamma^-_{1,r}(S_0 \times S_1) = \Gamma^-_{1,r}(S_0) + \Gamma^-_{1,r}(S_1)$$

nor that

$$\Gamma^+_{1,r}(S_0 \times S_1) = \Gamma^+_{1,r}(S_0) + \Gamma^+_{1,r}(S_1).$$

The construction of a counterexample follows the same spirit as that of Example 1.1.9 below.

Proof of Proposition 1.1.8:

The product of an ϵ-packing set for S_0 and of an ϵ-packing set for S_1 is an ϵ-packing set for $S_0 \times S_1$. This shows that

$$NP_\epsilon(S_0 \times S_1) \geq NP_\epsilon(S_0) \cdot NP_\epsilon(S_1)$$

and therefore

$$\frac{\log\log NP_\epsilon(S_0\times S_1)}{\log|\log\epsilon|}\geq\max\left(\frac{\log\log NP_\epsilon(S_0)}{\log|\log\epsilon|};\frac{\log\log NP_\epsilon(S_1)}{\log|\log\epsilon|}\right)$$

and

$$\frac{\log NP_\epsilon(S_0\times S_1)}{|\log\epsilon|^r}\geq\frac{\log NP_\epsilon(S_0)}{|\log\epsilon|^r}+\frac{\log NP_\epsilon(S_1)}{|\log\epsilon|^r}$$

from which follow

$$\Gamma^-_{2,1}(S_0\times S_1)\geq\max\Big(\Gamma^-_{2,1}(S_0);\Gamma^-_{2,1}(S_1)\Big)$$

and

$$\Gamma^-_{1,r}(S_0\times S_1)\geq\Gamma^-_{1,r}(S_0)+\Gamma^-_{1,r}(S_1).$$

The product of an ϵ-covering set for S_0 and of an ϵ-covering set for S_1 is an ϵ-covering set for $S_0\times S_1$. This shows that

$$NC_\epsilon(S_0\times S_1)\leq NC_\epsilon(S_0)\cdot NC_\epsilon(S_1)$$

and therefore

$$\frac{\log\log NC_\epsilon(S_0\times S_1)}{\log|\log\epsilon|}\leq$$

$$\leq\max\left(\frac{\log\log NC_\epsilon(S_0)}{\log|\log\epsilon|};\frac{\log\log NC_\epsilon(S_1)}{\log|\log\epsilon|}\right)+\frac{\log 2}{\log|\log\epsilon|}$$

and

$$\frac{\log NC_\epsilon(S_0\times S_1)}{|\log\epsilon|^r}\leq\frac{\log NC_\epsilon(S_0)}{|\log\epsilon|^r}+\frac{\log NC_\epsilon(S_1)}{|\log\epsilon|^r}$$

from which follow

$$\Gamma^+_{2,1}(S_0\times S_1)\leq\max\Big(\Gamma^+_{2,1}(S_0);\Gamma^+_{2,1}(S_1)\Big)$$

and therefore

$$\frac{\log\log NP_\epsilon(S_0\times S_1)}{\log|\log\epsilon|}\geq \max\left(\frac{\log\log NP_\epsilon(S_0)}{\log|\log\epsilon|};\frac{\log\log NP_\epsilon(S_1)}{\log|\log\epsilon|}\right)$$

and

$$\frac{\log NP_\epsilon(S_0\times S_1)}{|\log\epsilon|^r}\geq\frac{\log NP_\epsilon(S_0)}{|\log\epsilon|^r}+\frac{\log NP_\epsilon(S_1)}{|\log\epsilon|^r}$$

from which follow

$$\Gamma^-_{2,1}(S_0\times S_1)\geq\max\Big(\Gamma^-_{2,1}(S_0);\Gamma^-_{2,1}(S_1)\Big)$$

and

$$\Gamma^-_{1,r}(S_0\times S_1)\geq\Gamma^-_{1,r}(S_0)+\Gamma^-_{1,r}(S_1).$$

The product of an ϵ-covering set for S_0 and of an ϵ-covering set for S_1 is an ϵ-covering set for $S_0\times S_1$. This shows that

$$NC_\epsilon(S_0\times S_1)\leq NC_\epsilon(S_0)\cdot NC_\epsilon(S_1)$$

and therefore

$$\frac{\log\log NC_\epsilon(S_0\times S_1)}{\log|\log\epsilon|}\leq$$

$$\leq\max\left(\frac{\log\log NC_\epsilon(S_0)}{\log|\log\epsilon|};\frac{\log\log NC_\epsilon(S_1)}{\log|\log\epsilon|}\right)+\frac{\log 2}{\log|\log\epsilon|}$$

and

$$\frac{\log NC_\epsilon(S_0\times S_1)}{|\log\epsilon|^r}\leq\frac{\log NC_\epsilon(S_0)}{|\log\epsilon|^r}+\frac{\log NC_\epsilon(S_1)}{|\log\epsilon|^r}$$

from which follow

$$\Gamma^+_{2,1}(S_0\times S_1)\leq\max\Big(\Gamma^+_{2,1}(S_0);\Gamma^+_{2,1}(S_1)\Big)$$

and

$$\Gamma^+_{1,r}(S_0 \times S_1) \leq \Gamma^+_{1,r}(S_0) + \Gamma^+_{1,r}(S_1).$$

The first part of the proposition follows; the second one is similar.

$\boxed{1.1.8}$

1.1.5 An Example

An overoptimistic person might think that $\Gamma_{n,r}$ would always be defined. The following example should dispel such false hopes. This example deals only with $n = 1$ or $n = 2$, $r = 1$ but the generalization is obvious.

Example 1.1.9 *For any $0 < r^- \leq r^+ \leq +\infty$, let $0 \leq a \leq +\infty$ and $0 \leq b \leq +\infty$ be arbitrary except that if $r^- = r^+$ we must have $a \leq b$. A Cantor space K can be constructed with $\Gamma^-_{2,1}(K,d) = r^-$ and $\Gamma^+_{2,1}(K,d) = r^+$. $\Gamma^-_{1,r^-}(K,d) = a$ and $\Gamma^+_{1,r^+}(K,d) = b$.*

Make K the space of functions f from $\mathbb{N}$ to $\mathbb{N}$ with the property $f(n) < \varphi(n)$ where φ is a function to be defined. We define

$$d(f,g) = \exp(-\min\{n | f(n) \neq g(n)\}).$$

(Notice that for us $\mathbb{N}$ includes 0.) It is easy to see that

$$NP_\epsilon(K,d) = \prod_{i \leq -\log \epsilon} \varphi(i).$$

We may as well stick to the packing situation. The observations above give us

$$\log NP_\epsilon = \sum_{i \leq -\log \epsilon} \log \varphi(i).$$

Plugging this into the definition of $\Gamma_{2,1}$ we have

$$\Gamma^{+}_{2,1}(S,d) = \limsup_{n\to+\infty} \frac{1}{\log n} \log\left(\sum_{i\le n} \log \varphi(i)\right),$$

$$\Gamma^{-}_{2,1}(S,d) = \liminf_{n\to+\infty} \frac{1}{\log n} \log\left(\sum_{i\le n} \log \varphi(i)\right),$$

$$\Gamma^{+}_{1,r}(S,d) = \limsup_{n\to+\infty} \frac{1}{n^r} \sum_{i\le n} \log \varphi(i),$$

and

$$\Gamma^{-}_{1,r}(S,d) = \liminf_{n\to+\infty} \frac{1}{n^r} \sum_{i\le n} \log \varphi(i).$$

Now it is only a question of choosing the right φ; this is an easy task.

1.2 Spaces of Functions

1.2.1 The Metric

Let us apply the concepts of the previous section to spaces of functions. We will denote the open unit disc in the complex plane by Δ. The closed unit disc will be denoted by $\bar{\Delta}$ and the unit circle by $\partial\Delta$ or S^1. The space of all holomorphic functions defined on Δ will be denoted by $\mathcal{H}$; the subspace of bounded holomorphic functions defined on Δ will be denoted by $\mathcal{H}^\infty$; the subspace of holomorphic functions from Δ to Δ will be denoted by $\mathcal{M}_0$ and its closure, the space of holomorphic functions from Δ to $\bar{\Delta}$ will be denoted by $\mathcal{M}$. Notice that the only difference between these two spaces is that $\mathcal{M}$ contains the constant functions of unit norm.

Definition 1.2.1 *Let $A \subseteq \Delta$, $A \neq \emptyset$, and $f_1, f_2 \in \mathcal{H}$. For $n \geq 0$, we define*

$$d_A(f_1, f_2) = \sup_{z \in A} |f_1(z) - f_2(z)|.$$

This function can assume the value $+\infty$ when considered as defined on the whole of $\mathcal{H}$ but is finite if restricted to $\mathcal{H}^\infty$.

What we want to do is study this metric and others to be defined later, with special emphasis on their geometric properties.

We list some trivial consequences of the definition as a proposition.

Proposition 1.2.2 *The following are properties of d_A.*

(i) For any $A \subseteq \Delta$ and any $n \geq 0$, d_A is a pseudometric on M with $0 \leq d_A \leq 2$.

(ii) If $A \subseteq B$, $d_A \leq d_B$.

(iii) $d_A = d_{\bar{A}} = d_{\partial A}$.

(iv) If $A \subseteq \Delta$ has an accumulation point in Δ, d_A is a metric.

(v) If φ is a conformal bijection from Δ to itself, it induces an isomorphism between the two metric spaces $(\mathcal{M}, d_A)$ and $(\mathcal{M}, d_{\varphi(A)})$.

Proof of Proposition 1.2.2:

(i) and *(ii)* are immediate. *(iii)* follows from continuity and the Maximum Principle. *(iv)* follows from the fact that the set of zeroes of an analytic function can not have accumulation points. *(v)* follows from composition with φ.

$\boxed{1.2.2}$

A slightly less trivial proposition is the following.

Proposition 1.2.3 *The metric space $(\mathcal{M}, d_A)$ is compact.*

Proof of Proposition 1.2.3:

This is a well known fact. The proof will be omitted.

$\boxed{1.2.3}$

A question we want to ask is roughly how large d_A is, or, in other words, how large $\mathcal{M}$ is from the point of view of d_A.

1.2.2 Bounded and Unbounded Sets

The next proposition tells us that there is a big difference between sets that keep away from $\partial\Delta$ and those who don't.

Proposition 1.2.4 *For any $A \subseteq \Delta$, the conditions below are equivalent:*

(i) A does not accumulate on $\partial\Delta$.

(ii) $NP_\epsilon(\mathcal{M}, d_A) < +\infty$ for some $0 < \epsilon < 2$.

(iii) $NP_\epsilon(\mathcal{M}, d_A) < +\infty$ for all $0 < \epsilon < 2$.

(iv) $NC_\epsilon(\mathcal{M}, d_A) < +\infty$ for some $0 < \epsilon < 1$.

(v) $NC_\epsilon(\mathcal{M}, d_A) < +\infty$ for all $0 < \epsilon < 1$.

Proof of Proposition 1.2.4:

First consider the case where A does not accumulate on $\partial\Delta$. If we think of Δ as the hyperbolic plane, this means that A is bounded. For any ϵ we can find a finite family of balls of hyperbolic radius $\epsilon/3$ that will cover A. If the value of a function in $\mathcal{M}$ is given up to a precision of $\epsilon/3$ in the hyperbolic center of each of these balls, by the Schwartz-Pick theorem the function is defined up to an error of less than ϵ at any point of A. This gives us a finite covering of $\mathcal{M}$ by balls of radius ϵ in the metric d_A. We just proved that $NC_\epsilon(\mathcal{M}, d_A) < +\infty$ for all $0 < \epsilon < 1$. From proposition 1.1.2, it follows that $NP_\epsilon(\mathcal{M}, d_A) < +\infty$ for all $0 < \epsilon < 1$. We showed that *(i)* implies any of the other conditions.

We now consider the case when A does accumulate on $\partial\Delta$. We shall prove that for any $0 < \epsilon < 2$, $NP_\epsilon(\mathcal{M}, d_A) = +\infty$ by constructing an infinite ϵ-packing set. During the construction, ϵ will be fixed but arbitrary. Notice that the "hard" cases are those corresponding to ϵ large, close to 2. We will adopt the notation $\delta = 2 - \epsilon$. By proposition 1.1.2, it will follow that $NC_\epsilon(\mathcal{M}, d_A) = +\infty$ for all $0 < \epsilon < 1$. This will show that the negation of *(i)* implies the negation of any of the other conditions, establishing the proposition.

We might as well take an infinite strip around the real axis of width 2 as our domain and assume that A accumulates at $+\infty$. This means we can take a sequence $z_n = a_n + b_n i$ of points in A such that a_n goes to $+\infty$; we can assume that $a_{n+1} - a_n > C$ for a positive constant C to be specified. In the process of constructing the packing set, we shall use the function $f_k(z) = 2(kz)^{-2}\sin^2(kz) - 1$, where k is a positive real number, as a building block. Notice that for real z, the function assumes values between -1 and 1. On the strip of width 2 it will assume values in a neighborhood of this interval which can be made arbitrarily small by choosing a sufficiently small k. Define

$$B_k = \left(\sup_{|\mathrm{Im}(z)|<1} |f_k(z)| \right)^{-1}$$

and choose k such that $B_k > 1 - \delta/4$. Now notice that as the real part of z tends to $+\infty$, $f_k(z)$ tends to -1. Take C such that $\mathrm{Re}(z) > C$ implies $|f_k(z)+1| < \delta/4$. We are now ready to construct our infinite packing set: define $g_n(z) = B_k f_k(z - a_n)$. It is now straightforward to check that the set of all g_n is an ϵ-packing set.

$\boxed{1.2.4}$

In light of the previous two propositions, we are going to restrict our attention to the case $A \subseteq \Delta$, A compact.

1.2.3 Hyperdimension and Generalized Dimension

The Hausdorff dimension of $(\mathcal{M}, d_A)$ is going to be infinite in most cases, and it is a most astonishing coincidence that the concepts introduced in the last section turn out to be exactly what we need.

Definition 1.2.5 *If $A \subseteq \Delta$, we define*

$$\gamma^+_{n,r}(A) = \Gamma^+_{n,r}(\mathcal{M}, d_A), \qquad \gamma^-_{n,r}(A) = \Gamma^-_{n,r}(\mathcal{M}, d_A),$$

We drop the $+$ and $-$ signs when they are superfluous, i.e., when the corresponding limit exists.

The explanation concerning the intuitive meaning of $\Gamma_{n,r}$ in the previous section should also give an idea of what $\gamma_{n,r}$ means. $\gamma_{n,r}$ measures d_A, which amounts to measuring A, in some indirect and non-obvious way. This way of measuring A will of course be different for different values of n and r. We are going to see that at least in some cases $\gamma_{n,r}$ has a strong geometric meaning.

It is natural to ask whether the following conjecture is true.

Conjecture 1.2.6 *$\gamma_{n,r}(A)$ is always defined.*

This will turn out to be trivial in many cases (including $n \geq 3$) and will follow from our results in some others but the general case is hard and is not known to be true. (The unknown cases are $n = 1$, $1 < r < 2$ and $n = 2$, $r = 1$.)

Next we state some easy consequences of proposition 1.2.2 and definition 1.2.5.

Proposition 1.2.7

(i)

$$\gamma_{2,1}(\{0\}) = 1, \qquad \gamma_{1,1}(\{0\}) = 2.$$

(ii) If $A \subseteq B$ then (for any r)

$$\gamma^-_{n,r}(A) \leq \gamma^-_{n,r}(B), \qquad \gamma^+_{n,r}(A) \leq \gamma^+_{n,r}(B).$$

(iii)

$$\gamma^-_{n,r}(A) = \gamma^-_{n,r}(\bar{A}) = \gamma^-_{n,r}(\partial A),$$

$$\gamma^+_{n,r}(A) = \gamma^+_{n,r}(\bar{A}) = \gamma^+_{n,r}(\partial A).$$

(iv)

$$\gamma^+_{2,1}(A \cup B) \leq \max(\gamma^+_{2,1}(A); \gamma^+_{2,1}(B));$$

$$\gamma^+_{1,r}(A \cup B) \leq \gamma^+_{1,r}(A) + \gamma^+_{1,r}(B).$$

(v) If $A_i, 0 \leq i < n$ is a family of isometric sets (with respect to the Poincaré Metric) then

$$\gamma^-_{2,1}(\bigcup A_i) = \gamma^-_{2,1}(A_0),$$

$$\gamma^-_{1,r}(\bigcup A_i) \leq n\gamma^-_{1,r}(A_0).$$

(vi) If A is a finite set,

$$\gamma_{2,1}(A) = 1, \qquad \gamma_{1,1}(A) = 2|A|.$$

Proof of Proposition 1.2.7:

(i) is follows from the fact that $(\mathcal{M}, d_{\{0\}})$ is essentially isomorphic to $(\Delta, ||)$, meaning it is isomorphic once you identify points which are at zero distance. The disc is known to have Hausdorff dimension 2.

(ii) and *(iii)* are obvious consequences of the corresponding items of proposition 1.2.2.

From $d_{A\cup B} = \max(d_A, d_B)$ it follows that $(\mathcal{M}, d_{A\cup B})$ is a subspace of $(\mathcal{M}, d_A) \times (\mathcal{M}, d_B)$, and therefore has smaller $\Gamma^+_{2,1}$ and $\Gamma^+_{1,r}$. This observation together with proposition 1.1.8 gives us *(iv)*. *(v)* follows similarly from proposition 1.1.8 since now the spaces $(\mathcal{M}, d_{A_i})$ are all isomorphic.

(vi) is proved by induction from *(i)* and *(iv)*.

$\boxed{1.2.7}$

Chapter 2

The Conductivity Theorem

2.0 Introduction

The main theorem of this chapter is the Conductivity Theorem below.

Theorem 2.0.1 (Conductivity Theorem) *For any $A \subseteq \Delta$,*

$$\gamma_{1,2}(A) = \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

We already defined the left hand side of this equation. In the next section we proceed to define the right hand side, which is the conductivity (whence the name of the theorem). We then prove the special case of A connected and then finally the general case.

After writing this chapter, the author found out (thanks to the help of Curt McMullen) about the very closely related work of Erohin (see [2, 3]). Erohin proves what we call the

Conductivity Theorem for A connected and generalizations in directions different from ours (such as higher dimensions); we shall not deal with these generalizations here. Notice also that Erohin's proof is essentially the same as ours, only in a somewhat different language.

2.1 Conductivity and Energy

2.1.1 Conductivity

In what follows, all sets will be subsets of the complex plane. Most or all of what will be said can be generalized to other contexts; we will refrain from doing so, however, since it would be quite useless for us.

Definition 2.1.1 *Let A be an open set and let f be a harmonic function defined on A. $\mathbf{H}_f$ is a map from $\pi_1(A)$ (or $H_1(A)$) into $\mathbb{R}$ defined as follows:*

$$\mathbf{H}_f([\gamma]) = \int_\gamma \nabla f \wedge d\gamma.$$

H *is called the* holonomy *of f.*

It is obvious that $\mathbf{H}_f$ is a group homomorphism where $\mathbb{R}$ has the additive group structure. $\mathbf{H}_f$ measures the extent to which the harmonic conjugate of f fails to be well defined.

Definition 2.1.2 *Let A be an open set and B, C be a partition of its boundary. Let ϕ be the harmonic function defined on A with values 0 at B and 1 at C. ϕ will be called the* potential function.

We define $\mathbf{C}(A; B, C)$ *to be the absolute value of the flux of* $\nabla\phi$ *through a curve contained in* A *separating* B *from* C*; we call this the* conductivity *of* A *between* B *and* C.

First of all, let us notice that the above mentioned flux is independent of the choice of the curve since $\nabla\phi$ is closed. The exact meaning of saying that ϕ assumes certain values in certain subsets of the boundary would in general require some comments and some attention to technicalities. We will only talk about 'nice' cases though, and we therefore feel free to omit any such discussion.

The reason for calling this concept "conductivity" and using the symbol $\mathbf{C}$ should be clear if we think of ϕ as the electric potential, of $\nabla\phi$ as the flux of charge and of the flux of $\nabla\phi$ through an appropriate curve as the current. We shall sometimes use the symbol ℧ interchangeably with $\mathbf{C}$. Remember that by Ohm's law the conductivity is numerically equal to the intensity of the current when the difference of potential is one.

An important observation is that $\mathbf{C}$ is invariant by conformal maps.

Notice that this concept is interesting only when B and C are unions of connected components of ∂A; otherwise B and C "touch", we have a short-circuit and the conductivity becomes infinite. In the interesting case the two definitions above are related by the following lemma.

Lemma 2.1.3 *If* γ *is a closed curve separating* B *from* C *we have*

$$\mathbf{C}(A; B, C) = |\mathbf{H}_\phi(\gamma)|$$

where ϕ *is the potential function.*

Proof of Lemma 2.1.3:

Easy.

$\boxed{2.1.3}$

Example 2.1.4 *If A is the region between two concentric circles B and C of radii r and R then*

$$\mathbf{C}(A; B, C) = \frac{2\pi}{|\log R/r|}.$$

This can be shown directly since in this case we know exactly what ϕ is.

This is a special case of A being an annulus and B and C the connected components of its boundary. The general case can be transformed to this one by a conformal map. Since the Kobayashi metric is also invariant by conformal maps, we have:

Proposition 2.1.5 *Let A be an annulus and B and C the connected components of its boundary. Let d be the length of the unique simple closed geodesic in A with its Kobayashi metric. Then:*

$$\mathbf{C}(A; B, C) = d.$$

Proof of Proposition 2.1.5:

Above.

$\boxed{2.1.5}$

Let us conclude this subsection with a result concerning harmonic functions other than the potential.

Proposition 2.1.6 *Let A, B and C be as in the definition above. Let η be a harmonic function defined on A and extended to B and C. Then, if F is the flux of $\nabla\eta$ from B to C and $\mathbf{C}$ is the conductivity (as defined above)*

$$\mathbf{C} \cdot (\inf_{z \in C} \eta(z) - \sup_{z \in B} \eta(z)) \leq F \leq \mathbf{C} \cdot (\sup_{z \in C} \eta(z) - \inf_{z \in B} \eta(z)).$$

Proof of Proposition 2.1.6:

Easy.

$\boxed{2.1.6}$

2.1.2 Energy

There is another important concept called the *energy* of a map which we now introduce.

Definition 2.1.7 *If f is a real valued function defined on $D \subseteq \mathbb{C}$ we define the* energy *of f to be*

$$\mathbf{E}(f) = \frac{1}{2} \int_D |\nabla f|^2 dA.$$

Returning to the analogy with electricity, if f is the electric potential and ∇f is the flow of electric charge then $\mathbf{E}(f)$ will be one half of the amount of electric energy transformed into heat because of the resistence. In this interpretation the factor $1/2$ seems inappropriate; we keep it, however, since this is the usual definition and changing it would bring confusion. This observation makes the following proposition almost obvious.

Proposition 2.1.8 *Let $A \subseteq \mathbb{C}$ be an open set and B, C a partition of its boundary. Let ϕ be the harmonic function defined on A with values 0 at B and 1 at C. Then*

$$\mathbf{E}(\phi) = \frac{1}{2}\mathbf{C}(A; B, C).$$

Proof of Proposition 2.1.8:

This is an easy computation.

$\boxed{2.1.8}$

We have the following corollary.

Corollary 2.1.9 *Let $A \subseteq \Delta$ be a compact set. Let ϕ_A be the smallest superharmonic function satisfying $\phi_A(z) \geq 0$ for all z and $\phi_A(z) \geq 1$ for $z \in A$. Then we have*

$$\mathbf{E}(\phi_A) = \frac{1}{2}\mathbf{C}(\Delta - A; S^1, \partial A).$$

Proof of Corollary 2.1.9:

This is a consequence of the previous proposition on the domain $\Delta - A$.

$\boxed{2.1.9}$

Notice that this says that the Conductivity Theorem might just as well have been stated as

$$\gamma_{1,2}(A) = \frac{1}{\pi}\mathbf{E}(\phi_A)$$

where ϕ_A is defined as in the corollary.

2.1.3 Conductivity: Results

We now proceed to prove some further results about $\mathbf{C}$.

Definition 2.1.10 *For any $f \in \mathcal{H}$ we define* $\deg(f)$, *the* degree *of f, to be the number of zeroes of f in Δ counted with multiplicities.*

Remember that Δ is open, which means that zeroes on the boundary are not counted. The degree is either a non-negative integer or infinity. When f can be continuously extended to $\bar{\Delta}$, it is easy to see that $\deg(f)$ is the degree of $\tilde{f}(z) = f(z)/|f(z)|$ as a function from S^1 to itself.

Definition 2.1.11 *Let $W = (w_i), i \in I$ be a finite family of points of Δ. We define*

$$\beta_W(z) = \prod_{i \in I} \frac{z - w_i}{z\bar{w}_i - 1}.$$

We also define

$$E(W) = d_A(0, \beta_W).$$

It is easy to see that $deg(\beta_W) = |W|$, where $|W| = |I|$.

Lemma 2.1.12 *Let $A \subseteq \Delta$ be compact and let W be a family of n point of A. Let $f = \beta_W$ and let*

$$\frac{1}{|\log R|} = \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

Then

$$E(W) \geq R^n.$$

Proof of Lemma 2.1.12:

Consider the function $\phi_0 = \mathrm{Re}(\log f(z))$; it is well defined and continuous on $\Delta - A$, it is harmonic, its value is zero on S^1 and at most $\log(E(W))$ on ∂A. The flux of $\nabla\phi_0$ shall therefore be (in absolute value) at least $\mathbf{C}|\log(E(W))|$.

Now look at the harmonic conjugate of ϕ_0, which is

$$\phi_1 = \mathrm{Im}(\log f(z));$$

this function is *not* well defined on $\Delta - A$ since by going around once its value increases by $2\pi n$. It can easily be seen that this has also to be the flux. Therefore we have:

$$2\pi n \geq \mathbf{C}|\log(E(W))|$$

which proves the lemma.

$\boxed{2.1.12}$

Proposition 2.1.13 *Let $A \subseteq \Delta$ be arbitrary and let*

$$\frac{1}{|\log R|} = \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

Suppose $0 < R < 1$. Then there exists a sequence (w_n) of points of A with the property that $E(W_n) \leq kR^n$ where k is a constant and W_n is the family of the first n points of the sequence (w_n).

This proposition, together with lemma 2.1.12, gives us an alternative definition of conductivity.

Proposition 2.1.14 *Let $A \subseteq \Delta$ be arbitrary and let*

$$r_n = \min_{W \subseteq A, |W|=n} E(W).$$

Then

$$\lim_{n \to +\infty} \frac{n}{|\log r_n|} = \frac{1}{2\pi} \mathbf{C}(\Delta - A; S^1, \partial A).$$

Proof of Proposition 2.1.14:

This is a consequence of lemma 2.1.12 and proposition 2.1.13.

$\boxed{2.1.14}$

Proof of Proposition 2.1.13:

Intuitively, what we have to do is to pick the points in such a way as to minimize $E(W_n)$. This corresponds to taking points uniformly distributed according to the flux of current. Since this is a known result, we shall feel free to omit the details.

$\boxed{2.1.13}$

2.2 The Connected Case

2.2.1 Statement of the Theorem

The main theorem of this section is the following.

Theorem 2.2.1 *For any $A \subseteq \Delta$, A connected,*

$$\gamma_{1,2}(A) = \frac{1}{2\pi} \mathbf{C}(\Delta - A; S^1, \partial A).$$

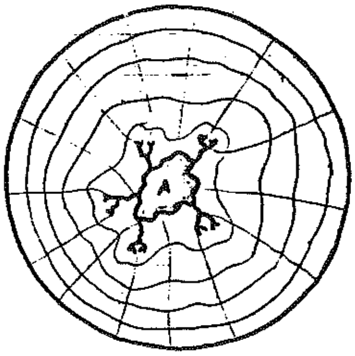

Figure 2.1: A and a sketch of ϕ_A (showing the level curves and the gradient trajectories).

Implicit in the statement is the fact that $\gamma_{1,2}(A)$ is defined, or that $\gamma_{1,2}^-(A) = \gamma_{1,2}^+(A)$. We shall concern ourselves with the case of A not necessarily connected later in this chapter.

We know this to be true for A a finite set, and, in a certain sense, for A accumulating on S^1. We shall suppose from now on that A is compact and that each connected component is simply connected. A special case where the theorem can be proven with relative ease is the case of A a closed disc.

Before we prove the theorem, let us note some of its easy but interesting consequences. These are in fact consequences of the special case where A a closed disc.

Corollary 2.2.2 *If $A \subseteq \Delta$ is bounded away from the boundary then*

$$\gamma_{1,2}^+(A) < +\infty.$$

If A is any compact non-empty set, we have $1 \leq \gamma_{2,1}^-(A) \leq \gamma_{2,1}^+(A) \leq 2$.

Proof of Corollary 2.2.2:

Just observe that A is contained in some big disc.

$\boxed{2.2.2}$

Corollary 2.2.3 *If $A \subseteq \Delta$ has non-empty interior, $\gamma^-_{1,2}(A) > 0$. Also, if A connects two distinct points the same conclusion holds. In any of these cases, assuming A compact, $\gamma_{2,1}(A) = 2$.*

Proof of Corollary 2.2.3:

If A has non-empty interior, it contains a disc and the result follows. If A connects two points, the union of a finite number of sets obtained from A by rigid motions in the hyperbolic plane will contain the boundary of a set of non-empty interior, and will therefore have positive $\gamma^-_{1,2}$. The result follows from proposition 1.2.7.

$\boxed{2.2.3}$

Notice that this already tells us the value of $\gamma_{2,1}$ for all but infinite totally disconnected sets. The reason why $\gamma_{2,1}$ was so much easier to compute than $\gamma_{1,r}$ is that $\gamma_{2,1}$ is much less sensitive.

We shall use the following lemma in the proof of theorem 2.2.1. Let us anticipate that, when A is a disc, we shall have $\varphi_n(z) = z^n$. This special case should be kept in mind when reading the proof of the theorem. It also makes the lemma somewhat more natural, since what it says is that an analogue to z^n exists for sets other than the disc.

Lemma 2.2.4 *Let $A \subseteq \Delta$ be any connected compact set and let R be defined by:*

$$\frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A) = \frac{1}{|\log R|}.$$

Then there is a family $\varphi_n, n \in \mathbb{N}$, of functions from Δ to $\mathbb{C}$ with the following properties:

(i) Any bounded function $f : \Delta \to \mathbb{C}$ can be written uniquely as a series $f = \sum a_n \varphi_n$. The series converges uniformly and absolutely on any compact subset of Δ.

(ii) $\varphi_0 = 1$ and $\deg(\varphi_n) = n$.

(iii) $|\varphi_n(z)| < K$ for all $z \in \Delta$ and all n, where K is a constant.

(iv) For any $f = \sum a_n \varphi_n \in \mathcal{M}$, $|a_n < 1|$.

(v) For any $R' > R$, there exists a constant $K_{R'}$ such that $d_A(0, \varphi_n) \leq K_{R'} R'^n$.

(vi) If $d_A(0, f) \leq \epsilon$ then $|a_n| \leq \epsilon K R^{-n}$, where K is a constant.

The proof of this lemma will be postponed. Let us note that for A a disc and $\varphi_n(z) = z^n$ the lemma indeed holds. The only items which are not entirely obvious are *(iv)* and *(vi)*; these follow from an application of the Cauchy Integral Formula.

Let us divide the theorem into two parts which have independent proofs anyway.

Lemma 2.2.5 *For any $A \subseteq \Delta$, A connected,*

$$\gamma_{1,2}(A) \geq \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

Lemma 2.2.6 *For any $A \subseteq \Delta$, A connected,*

$$\gamma_{1,2}(A) \leq \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

Proof of Theorem 2.2.1:

This is an obvious consequence of lemmas 2.2.5 and 2.2.6.

$\boxed{2.2.1}$

2.2.2 Proof of Lemma 2.2.5

Proof of Lemma 2.2.5:

The idea behind this proof is really simple: all we have to do is build a large enough packing set.

We are going to express the functions in terms of the series expansion given by the lemma. A typical element $a \in \mathcal{M}$ will be written as

$$a(z) = \sum a_n \varphi_n(z).$$

The annoying feature of this way of writing functions is that it is not always easy to tell from the a_n whether or not a is in $\mathcal{M}$. This will not be too serious a problem, though, since $\gamma_{1,2}$ is not very sensitive.

When building a large ϵ-packing set, the first thing we have to worry about is that each point has to be in $\mathcal{M}$. Now, if (c_n) is a series of positive numbers with $\sum c_n \leq 1/K$ (where K is the constant mentioned in item *(iii)* of lemma 2.2.4) then the condition $|a_n| \leq c_n$ is clearly sufficient to guarantee that $a \in \mathcal{M}$.

Our next worry is to make sure the points in our set are not too close. Finding the exact value of d_A from the series would be too hard, but, again, we don't really need it; a rough estimate will suffice. By item *(vi)* of lemma 2.2.4, two functions are separated by a distance of at least ϵ provided the n-th coefficients in their power series differ from one another by at least $KR^{-n}\epsilon$.

Before going on, let us notice that we can fit at least $\lceil r/d \rceil^2$ in a disc of radius r keeping them a distance of at least d from each other.

Let us pick $c_n = (1-\theta)\theta^n/K$, where θ is some number between 0 and 1. From the above observation, we have at least

$$\left\lceil \frac{(1-\theta)\theta^n}{K^2R^{-n}\epsilon} \right\rceil^2$$

good values for a_n, in the sense that if we pick the a_n from among the good values we are guaranteed to have an ϵ-packing set.

This gives us the following estimate for NP_ϵ, where n is arbitrary:

$$NP_\epsilon \geq \prod_{0\leq i<n} \frac{(1-\theta)^2\theta^{2i}R^{2i}}{K^4\epsilon^2} = \frac{(1-\theta)^{2n}\theta^{n^2-n}R^{n^2-n}}{K^{4n}\epsilon^{2n}}$$

or

$$\begin{aligned} \log NP_\epsilon \quad &\geq \quad 2n|\log\epsilon| - 2n|\log(1-\theta)| - 4n\log K \\ &\quad -(n^2-n)|\log\theta| - (n^2-n)|\log R|. \end{aligned}$$

Now, of course, we want to choose the best value for n, which means we want to include in our product only the terms larger than 1. This corresponds to solving the "equation"

$$\frac{(1-\theta)^2\theta^{2n}R^{2n}}{K^4\epsilon^2} \approx 1$$

which gives

$$n \approx \frac{1}{|\log\theta| + |\log R|}|\log\epsilon| - \frac{2\log K + |\log(1-\theta)|}{|\log\theta| + |\log R|}.$$

What we want to do now is substitute this expression for n in our estimate of NP_ϵ. In order not to waste time and effort, however, we can remember that we are going to divide $\log NP_\epsilon$ by $\log^2 \epsilon$ and take the limit, so we may just as well compute the estimate for $\log NP_\epsilon$ and ignore terms smaller than $\log^2 \epsilon$. This will give us (after some simplifications)

$$\log NP_\epsilon \geq \frac{1}{|\log \theta| + |\log R|} \log^2 \epsilon + C_1 \log \epsilon + C_0,$$

where the exact values of C_0 and C_1 are irrelevant. From this inequality we get

$$\gamma_{1,2}^{-}(A) \geq \frac{1}{|\log \theta| + |\log R|}$$

and, since this holds for θ arbitrarily close to 1:

$$\gamma_{1,2}^{-}(A) \geq \frac{1}{|\log R|}.$$

This gives us half of the theorem.

$\boxed{2.2.5}$

2.2.3 Proof of Lemma 2.2.6

Proof of Lemma 2.2.6:

The idea behind this proof is as simple as the one behind the proof of lemma 2.2.5: now all we have to do is build a small enough covering set. When building covering sets we will allow ourselves to include "fake points", i.e., points which are not really in our space or may not even exist. By looking at the way the inequalities go we can see that this does not hurt the logic of the proof.

We are again going to express the functions in terms of the series expansion given by the lemma. A typical element $a \in \mathcal{M}$ will be written as

$$a(z) = \sum a_n \varphi_n(z).$$

Remembering that including "bad" points is no problem, we can say that each a_n has absolute value at most 1.

Take $R' > R$. Notice that the n-th term will contribute on A with an absolute value of at most $|a_n| K R'^n$. Let (c_n) be a series with $\sum c_n \leq 1$. Now if we pick each a_n from an ϵ_n-covering set of the unit disc, with $\epsilon_n = c_n \epsilon K^{-1} R'^{-n}$, we will have an ϵ-covering set of $\mathcal{M}$. This may look like it is going to be infinite, but if all but a finite number of the ϵ_n-covering sets have exactly one element we are safe. This can be done by choosing c_n such that $\epsilon_n \geq 1$ for almost every n. In other words, we want

$$c_n \geq \epsilon^{-1} K R'^n$$

for almost every n. The easiest way of doing this is picking some n_0 such that

$$\sum_{n \geq n_0} \epsilon^{-1} K R'^n \leq \frac{1}{2},$$

the existence of such an n_0 being trivial. Now we can define

$$c_n = \begin{cases} \frac{1}{2n_0}, & \text{if } n < n_0; \\ \epsilon^{-1} K R'^n, & \text{if } n \geq n_0. \end{cases}$$

This means that we have

$$\epsilon_n = \begin{cases} \frac{\epsilon K^{-1} R'^{-n}}{2n_0}, & \text{if } n < n_0; \\ 1, & \text{if } n \geq n_0. \end{cases}$$

We will need an estimate of n_0; since

$$\sum_{n \geq n_0} \epsilon^{-1} K R'^n = \epsilon^{-1} K R'^n (1 - R')^{-1}$$

we have that

$$n_0 = \left\lceil \frac{1}{|\log R'|} \Big(|\log \epsilon| + |\log(1 - R')| + \log K + \log 2 \Big) \right\rceil \approx \frac{|\log \epsilon|}{|\log R'|}$$

is the best choice for n_0. This gives us complete instructions for building our ϵ-covering set.

Before going on, let us prepare ourselves to meet a minor difficulty ahead. We would like our construction to imply that $\epsilon_n \leq 2$ for all n. This is, unfortunately, usually not true. Let us therefore take n_1 to be the greatest integer smaller than or equal to n_0 with the property that $\epsilon_n \leq 2$ for all $n \leq n_1$. We can easily get

$$n_1 = \min \left(n_0, \left\lfloor \frac{1}{|\log R'|} \Big(\log n_0 + \log K + \log 4 + |\log \epsilon| \Big) \right\rfloor \right) \approx \frac{|\log \epsilon|}{|\log R'|},$$

an estimate for n_1 which we will need later on.

Our next step will be to make an estimate of the size of the covering set. First of all we need to find out how many points we need to make an ϵ_n-covering set of the unit disc, and it is easy to see that $4\epsilon_n^{-2}$ suffice provided $\epsilon_n \leq 2$, which is true for $n \leq n_1$, and for the rest one point is sufficient. This gives us

$$NC_\epsilon \leq \prod_{i=0}^{n_1} \frac{16 n_0^2}{\epsilon^2 K^{-2} R'^{-2n}} = \frac{2^{4n_1} n_0^{2n_1} K^{2n_1} R'^{n_1^2 - n_1}}{\epsilon^{2n_1}}$$

or

$$\begin{aligned} \log NC_\epsilon \quad \leq \quad & 2n_1 |\log \epsilon| - (n_1^2 - n_1) |\log R'| + \\ & + 2n_1 \log K + 4n_1 \log 2 + 2n_1 \log n_0. \end{aligned}$$

Substituting in n_0 and n_1 and ignoring terms smaller than $\log^2 \epsilon$ we get

$$\log NC_\epsilon \leq \frac{1}{|\log R'|} \log^2 \epsilon + (\text{ smaller terms })$$

which gives us the estimate

$$\gamma^+_{1,2}(A) \leq \frac{1}{|\log R'|}.$$

Since R' can be made arbitrarily close to R, we have

$$\gamma^+_{1,2}(A) \leq \frac{1}{|\log R|}$$

which proves our lemma.

$\boxed{2.2.6}$

2.2.4 Proof of Lemma 2.2.4

Before we handle the general case of lemma 2.2.4, let us look at a special case.

Lemma 2.2.7 *Let $A \subseteq \Delta$ be a slit from $-D$ to $+D$, and let ψ be the bijection between K and $\Delta - A$ (see figure 2.2), where K is the circular annulus of inner radius R, with*

$$\frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A) = \frac{1}{|\log R|}.$$

Let also $w = \psi^{-1}(z)$, for $z \in \Delta - A$.

Then there is a family $\varphi_n, n \in \mathbb{N}$, of functions from Δ to $\mathbb{C}$ with the following properties:

(i) Any bounded function $f : \Delta \to \mathbb{C}$ can be written uniquely as a series $f = \sum a_n \varphi_n$.

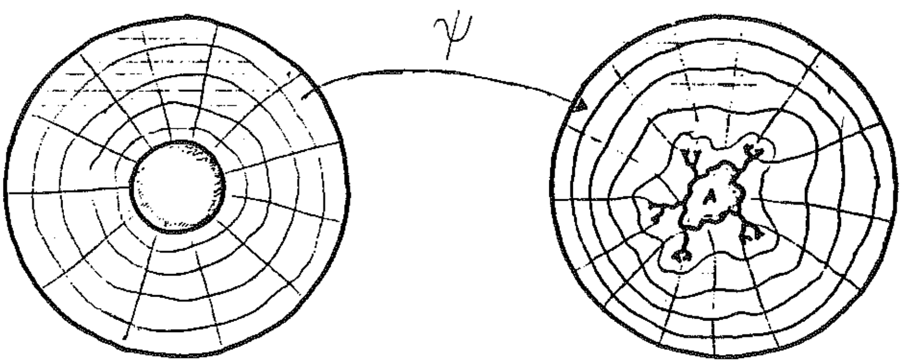

Figure 2.2: ψ is a bijection from K to $\Delta - A$.

(ii) We can write any function $f : \Delta \to \mathbb{C}$ as a Laurent series in w. If we write

$$\varphi_n(z) = \sum_{i \in \mathbb{Z}} b_i w^i$$

we have $b_n = 1$ and $b_i = 0$ for $i \leq 0, i \neq n$. In other words, we can write

$$\varphi_n(z) = w^n + \sum_{i<0} b_i w^i.$$

(iii) $|\varphi_n(z)| < 2$ for all $z \in \Delta$ and all n.

(iv) For any $f = \sum a_n \varphi_n \in \mathcal{M}$, $|a_n < 1|$.

(v) $d_A(0, \varphi_0) = 1$; $d_A(0, \varphi_n) = 2R^n$ for $n > 0$.

(vi) If $d_A(0, f) \leq \epsilon$ then $|a_n| \leq \epsilon R^{-1-n}$.

Proof of Lemma 2.2.7:

We start by transforming the annulus $\Delta - A$ into the annulus K bounded by the circles of radii R and 1, where R will be $\exp(-\frac{1}{2\pi}\mathrm{C}(\Delta - A; S^1, \partial A))$. Let us denote the map from K to $\Delta - A$ by ψ. For any $f \in \mathcal{M}$, $\tilde{f} = f \circ \psi$ is analytic on K and bounded in absolute value by 1. We can express $\tilde{f}$ in terms of Laurent series as follows:

$$\tilde{f}(w) = \cdots + a_{-2}w^{-2} + a_{-1}w^{-1} + a_0w^0 + a_1w^1 + a_2w^2 + \cdots.$$

There is a problem in considering the Laurent series of $\tilde{f}$ as a name for f, namely, most Laurent series will correspond to functions defined only on $\Delta - A$ and impossible to extend to the entire disc. We need therefore some information as to which series, or at least how many series, actually represent a legitimate f.

The first thing we observe is that for any $\tilde{f}$, f is defined in $\Delta - A$, therefore any Laurent series for $\tilde{f}$ will give us a Laurent series for f. Notice furthermore that this map from the space of Laurent series converging in K to that of all Laurent series (the image of which will in fact turn out to converge in an annulus bounded exteriorly by the unit circle) is linear and continuous. The "good" Laurent series are those whose image is in fact a Taylor series.

In our case, however, we are lucky enough to know exactly what Laurent series are good. These are the series that satisfy the condition $\tilde{f}(w) = \tilde{f}(\bar{w})$ for all w on the inner circle of radius R. In terms of the coefficients, this gives us the family of conditions:

$$a_nR^n = a_{-n}R^{-n}.$$

This means the "good" $\tilde{f}$ can be written as follows:

$$\tilde{f}(w) = a_0w^0 + a_1(w^1 + R^2w^{-1}) + a_2(w^2 + R^4w^{-2}) + \cdots$$

thus allowing us to write

$$f(z) = a_0\varphi_0(z) + a_1\varphi_1(z) + a_2\varphi_2(z) + \cdots$$

where $\varphi_0(z) = 1$, $\varphi_n(z) = (w^n + R^{2n}w^{-n})$ for $n > 0$ and $w = \psi^{-1}(z)$. This gives us the promised new series expansion, taking care of *(i)* and *(ii)*.

(iii) and *(v)* are straightforward computations. *(iv)* follows from the Cauchy integral formula on the circle of radius 1. *(vi)* follows from the Cauchy integral formula on the circle of radius R.

$\boxed{2.2.7}$

The following is a slightly stronger version of lemma 2.2.4, in order to stress the similarities with the special case of the slit.

Lemma 2.2.8 *Let $A \subseteq \Delta$ be any connected, simply connected, compact set and let ψ be the bijection between K and $\Delta - A$ (see figure 6.1), where K is the circular annulus of inner radius R, with*

$$\frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A) = \frac{1}{|\log R|}.$$

Let also $w = \psi^{-1}(z)$, for $z \in \Delta - A$.

Then there is a family $\varphi_n, n \in \mathbb{N}$, of functions from Δ to $\mathbb{C}$ with the following properties:

(i) Any bounded function $f : \Delta \to \mathbb{C}$ can be written uniquely as a series $f = \sum a_n\varphi_n$. The convergence is uniform on any compact subset of Δ.

(ii) We can write any function $f : \Delta \to \mathbb{C}$ as a Laurent series in w. If we write

$$\varphi_n(z) = \sum_{i\in\mathbb{Z}} b_i w^i$$

we have $b_n = 1$ *and* $b_i = 0$ *for* $i \leq 0, i \neq n$. *In other words, we can write*

$$\varphi_n(z) = w^n + \sum_{i<0} b_i w^i.$$

We have $\varphi_0(z) = 1$ *and* $\deg(\varphi_n) = n$.

(iii) $|\varphi_n(z)| < K$ *for all* $z \in \Delta$ *and all* n, *where* K *is a constant.*

(iv) For any $f = \sum a_n \varphi_n \in \mathcal{M}$, $|a_n < 1|$.

(v) For any $R' > R$, *there exists a constant* $K_{R'}$ *such that* $d_A(0, \varphi_n) \leq K_{R'} R'^n$.

(vi) If $d_A(0, f) \leq \epsilon$ *then* $|a_n| \leq \epsilon R^{-1-n}$.

We shall need the following lemma:

Lemma 2.2.9 *Let* S^2 *be the Riemann sphere. Let* $M, N \subseteq S^2$ *be two open topological discs such that* $M \cup N = S^2$ *and* $M \cap N$ *is an annulus.*

Then if f *is any holomorphic function defined on* $M \cap N$ *we can write* $f = f_M + f_N$ *where* f_M *is defined on* M *and* f_N *on* N. *This representation is unique up to an additive constant. Furthermore, if* f *is bounded above by* C *in absolute value, we can take* f_M *and* f_N *to be bounded above by* kC *in absolute value, where* k *is a constant (depending on* M *and* N *but independent of* f *).*

The conclusions above imply that

$$0 \longrightarrow H(S^2) \longrightarrow H(M) \oplus H(N) \longrightarrow H(M \cap N) \longrightarrow 0$$

is an exact sequence.

Proof of Lemma 2.2.9: .

Suppose first that the boundaries of M and N are smooth. Let us identify S^2 with the complex plane $\mathbb{C}$; we can suppose without loss of generality that 0 is in $M - \bar{N}$ and that ∞ is in $N - \bar{M}$. Let γ_0 be the boundary of M oriented counterclockwise and γ_1 be the boundary of N oriented clockwise. If z is in $M \cap N$ we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}dw.$$

Now we can make

$$f_M(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(w)}{w-z}dw$$

and

$$f_N(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}dw.$$

It is clear that f_M is defined in M and f_N is defined in N; the only possible doubt would be whether f_N is analytic around ∞, which it is (we have $f_N(\infty) = 0$). It is also clear that on $M \cap N$ we have $f = f_M + f_N$. The uniqueness of the solution (up to additive constant) is a consequence of the fact that the only holomorphic functions defined on the entire sphere are constants. The only thing left to prove is the estimate.

Let $M_0 \subseteq M$ and $N_0 \subseteq N$ be open discs satisfying $M_0 \cup N_0 = S^2$, $M_0 \cap N_0$ is an annulus, and, for some positive constant δ, $d(M_0, \partial M) > \delta$ and $d(N_0, \partial N) > \delta$, where $d(z_0, z_1) = |z_0 - z_1|$. Let L be an upper bound on the lengths of γ_0 and γ_1. If $|f(z)| < C$ for $z \in M \cap N$ it follows that, for $z \in M_0$, $|f_M(z)| < \frac{1}{\delta}LC$ and similarly for N_0 and f_N. It now follows easily that $k = \frac{1}{\delta}L + 1$ will satisfy the lemma.

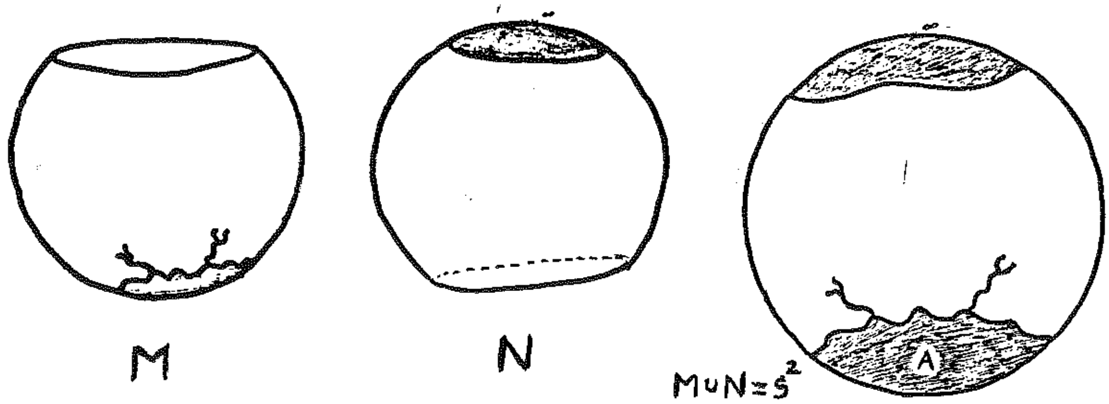

Figure 2.3: How to glue M and N and put the problem in a sphere.

In the case where the boundaries may are not smooth, we can just pick a smaller annulus, and the lemma will follow.

$\boxed{2.2.9}$

Proof of Lemma 2.2.8:

This proof starts the same way as the proof of lemma 2.2.7; we define K, ψ and the other concepts just like in the other proof. The difficulty here is that unlike the other case, we don't know exactly what the "good" $\tilde{f}$ are.

Let M be our original Δ in the z-sphere with A sitting in it. Let N be the exterior of the circle of radius R in the w-sphere, therefore containing K. We can use ψ to glue M and N to each other, thus giving us a topological sphere. We know from the theory of Riemann surfaces that all topological spheres with a complex analytic structure are isomorphic. Now take f to be the function w^n, which is indeed holomorphic on $M \cap N$ and bounded in absolute value by 1. We can now use our lemma in order to obtain f_M and f_N. Given what

N is, we can write

$$f_N(w) = a_0 + \sum_{i<0} a_i w^i.$$

We can suppose without loss of generality that $a_0 = 0$. We now have

$$f_M(w) = w^n + \sum_{i<0} a_i w^i.$$

We shall now take $\varphi_n = f_M$.

We already proved *(ii)* and *(iii)*; *(v)* follows from the observation that we can make M smaller, just a small neighbourhood of A, thus causing f to be bounded in absolute value by R'^n, where $R' > R$, without affecting f_M. In order to see that *(i)* is true, write

$$f(w) - \sum_{i \geq 0} a_i \varphi_i(w) = g(w).$$

Now it is easy to see that g is defined on both M and N, which means it has to be a constant and therefore 0. *(iv)* and *(vi)* are proven just like in lemma 2.2.7 by the Cauchy integral formula.

$\boxed{2.2.8}$

Proof of Lemma 2.2.4:

Obvious.

$\boxed{2.2.4}$

2.2.5 Why $\Delta - A$ Matters

This subsection is somewhat parenthetical. It addresses the question of why the complement of A is so important in determining $\gamma_{1,2}(A)$. We answer this question by constructing a natural bijection between $(\mathcal{H}^\infty(\Delta), d_{A_0})$ and $(\mathcal{H}^\infty(\Delta), d_{A_1})$ from a bijection between $\Delta - A_0$ and $\Delta - A_1$. This could all be said using the language of functors, but it hardly seems to help.

Let $A_0, A_1 \subseteq \Delta$ be two compact sets and let ψ be an analytic bijection between $\Delta - A_0$ and $\Delta - A_1$. Let us consider that A_0 and A_1 live in two different discs called respectively Δ_0 and Δ_1. Δ_0 lives inside a Riemann sphere; call the complement of A_0 in this sphere M_0. We can glue M_0 to Δ_1 using ψ to get a Riemann sphere. Since $\mathcal{H}^\infty(S^2) = \mathbb{C}$ we have the diagram below

$$\begin{tikzcd}
 & & \mathcal{H}^\infty(\Delta_0) \oplus \mathcal{H}^\infty(M_0) \arrow[dr] & & \\
0 \arrow[r] & \mathbb{C} \arrow[ur] \arrow[dr] & & \mathcal{H}^\infty(\Delta - A) \arrow[r] & 0 \\
 & & \mathcal{H}^\infty(\Delta_1) \oplus \mathcal{H}^\infty(M_0) \arrow[ur] & &
\end{tikzcd}$$

where the upper and the lower lines are exact. Let us see how this diagram gives us the promised bijection.

Start with a function f in $(\mathcal{H}^\infty(\Delta), d_{A_0})$. By restriction we can consider f to be also in $\mathcal{H}^\infty(\Delta - A)$. Looking at the diagram we see that this f is the image of a unique ordered pair of functions $(\tilde{f}; \hat{f})$ in $\mathcal{H}^\infty(\Delta_1) \oplus \mathcal{H}^\infty(M_0)$ provided we demand that $\hat{f}(\infty) = 0$. (Notice that ∞ is well defined by construction.) This gives us the required map. The proof that it is a bijection is easy.

The existence of such a bijection is in itself not very exciting: it is a bijection between a space and itself. The interesting thing about it is the way it behaves with respect to the metrics: we would like to prove that this map is a homeomorphism. This is, unfortunately, not true in general, but the next proposition shows that in a sense it is almost true.

Proposition 2.2.10 *Let A_0 and A_1 be as above and let the bijection described take f to $\tilde{f}$. Then if B_0 contains an open neighbourhood of A_0 and if $B_1 = A_1 \cup \psi(B_0)$ then there exist constants K_0 and K_1 with*

$$K_0 d_{B_0}(f,g) \leq d_{B_1}(\tilde{f},\tilde{g}) \leq K_1 d_{B_0}(f,g).$$

The constants depend on the choice of B_0.

Proof of Proposition 2.2.10:

Easy.

$\boxed{2.2.10}$

2.3 The General Case

2.3.1 Statement of the Theorem

In this section we prove the conductivity theorem in its general form. We state it once more in order to refresh the reader's memory.

Theorem 2.0.1 *For any $A \subseteq \Delta$,*

$$\gamma_{1,2}(A) = \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

The natural thing to expect at this point is that we shall be able to prove a corresponding generalization of lemma 2.2.4 and prove our theorem in a very similar way to that in which we proved theorem 2.2.1. In the next subsection we shall see that at least the most obvious generalizations do *not* work and why items *(v)* and *(vi)* specifically do not carry over to the new situation.

Let us now look at a simple case where things do work. This result is of course a special case of the Conductivity Theorem.

Proposition 2.3.1 *Let $A \subseteq \Delta$ be a compact set. Let $f(z) = z^n$ and let $A_n = f^{-1}(A)$. Then we have $\gamma_{1,2}(A_n) = n\gamma_{1,2}(A)$.*

Proof of Proposition 2.3.1:

We have to prove that $\Gamma_{1,2}(\mathcal{M}, d_{A_n}) = n\Gamma_{1,2}(\mathcal{M}, d_A)$. Let $\zeta = e^{\frac{2\pi i}{n}}$. For $0 \leq i < n$, define

$$\mathcal{M}_i = \{f \in \mathcal{M} | \forall z \in \Delta, f(\zeta z) = \zeta^i f(z)\}.$$

$\mathcal{M}_i$ corresponds to the space of functions whose Taylor series are of the form

$$f(z) = a_i z^i + a_{i+n} z^{i+n} + a_{i+2n} z^{i+2n} + \cdots = \sum_{j \equiv i \pmod{n}} a_j z^j.$$

From this it follows easily that

$$\mathcal{M} = \bigoplus_{0 \le i < n} \mathcal{M}_i$$

and therefore

$$\Gamma_{1,2}(\mathcal{M}, d_{A_n}) = \sum_{0 \le i < n} \Gamma_{1,2}(\mathcal{M}_i, d_{A_n})$$

(provided all of these exist). We shall prove our proposition by proving that $\Gamma_{1,2}(\mathcal{M}_i, d_{A_n}) = \Gamma_{1,2}(\mathcal{M}, d_A)$ for any value of i.

Let us first consider the case $i = 0$. In this case there is indeed an isomorphism between $(\mathcal{M}, d_A)$ and $(\mathcal{M}_0, d_{A_n})$, which takes f to $f \circ g$ where $g(z) = z^n$. This implies $\Gamma_{1,2}(\mathcal{M}_0, d_{A_n}) = \Gamma_{1,2}(\mathcal{M}, d_A)$. Let us denote by $\mathcal{M}_n$ that subspace of $\mathcal{M}_0$ of functions satisfying $f(0) = 0$. It is easy to check that $\Gamma_{1,2}(\mathcal{M}_n, d_{A_n}) = \Gamma_{1,2}(\mathcal{M}_0, d_{A_n})$.

Let us now define, for each i, $0 < i < n$, two maps θ_0 and θ_n. θ_0 is a map from $\mathcal{M}_0$ to $\mathcal{M}_i$ that takes $f(z)$ to $z^i f(z)$. This is a bijection, and it is distance decreasing. This implies that

$$\Gamma_{1,2}(\mathcal{M}_0, d_{A_n}) \le \Gamma_{1,2}(\mathcal{M}_i, d_{A_n}).$$

θ_n is a map from $\mathcal{M}_i$ to $\mathcal{M}_n$ that takes $f(z)$ to $z^{(n-i)} f(z)$. This is a bijection, and it is distance decreasing. This implies that $\Gamma_{1,2}(\mathcal{M}_i, d_{A_n}) \le \Gamma_{1,2}(\mathcal{M}_n, d_{A_n})$. These two inequalities combined easily imply the equality we need and therefore the proposition.

$\boxed{2.3.1}$

The correct proof of the Conductivity Theorem to be presented later will be in some sense a generalization of both Theorem 2.2.1 and Proposition 2.3.1.

2.3.2 What goes Wrong with the Old Proof

As we mentioned in the previous subsection, the natural thing to expect at this point is that we shall be able to prove a corresponding generalization of lemma 2.2.4 and prove our theorem in a very similar way to that in which we proved theorem 2.2.1. Let us see what happens when we try to generalize the proof of lemma 2.2.4.

The first difficulty here is that ψ can not be defined in the same way we used in the case of A connected since $\Delta - A$ is no longer an annulus. There is no reason to panic, however. ϕ, the potential function, is defined in all cases. In the case where A is connected, all level curves of ϕ are topological circles which are images under ψ of actual round circles in K. In the general case, not all level curves of ϕ are topological circles, but those level curves close enough to S^1 are. We can define ψ from a thin circular corona inside S^1 to the annulus contained between S^1 and some level curve. This new definition of ψ is just like the old one, except that the domain where it is defined is possibly smaller, which may seem not to be too serious a problem since we can make analytic continuation. This allows us to talk about Laurent series in w and to prove most of the lemma exactly as we proved lemma 2.2.8. The only items where the proof does not carry over directly are *(v)* and *(vi)*.

In fact, it is particularly hard to carry the proof of these items over to the new situation, the reason being that they are false. Let us see why this is so before going on. In order to simplify the discussion, let us consider the situation where A has two connected components A_0 and A_1. In the discussion below, the reader should have in mind figure 2.4.

Let us consider how we would go about defining $\varphi_n(z)$. Remember that ϕ is defined on

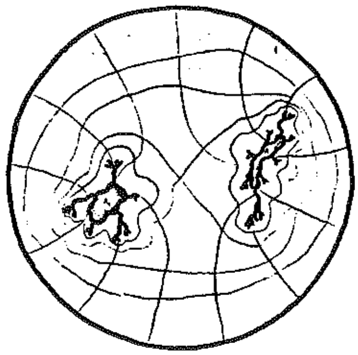

Figure 2.4: A and a sketch of ϕ_A; A has two components.

$\Delta - A$ and that w is defined only on a neighbourhood of S^1, more exactly, in the region outside the '8' shaped level curve of ϕ which passes through the point where the gradient of ϕ equals 0. ψ is also defined outside the '8' and by gluing Δ to a neighbourhood of ∞ in the w plane we can see that the function w^n can be decomposed uniquely in the form $w^n = \varphi_n(z) - \sum_{i<0} a_i w_i$, where $\varphi_n(z)$ is defined on Δ. This is of course 'the same' definition as the one we used in lemma 2.2.8. The question is: are these functions as well behaved in the situation where A is disconnected as they were shown to be in the connected case? As we already mentioned, the answer is no, and the source of the problems is the fact that w^n is usually not defined on all of $\Delta - A$. It is clearly defined in the region outside the '8' curve, and indeed if we take away from $\Delta - A$ any curve connecting A_0 to A_1, w^n will be defined in what is left. When you approach the curve from the two sides, however, you usually do not get the same limit. Since ϕ determines the absolute value of w, however, we know that the function on the two sides of the curve will differ only by multiplication by a fixed complex number of absolute value 1. If this number happens to be equal to 1, then w^n is indeed defined on all of $\Delta - A$, but otherwise it just can not be extended in this way. If for the case $n = 1$ we call this number ω the corresponding number for arbitrary n will be ω^n.

The above discussion should motivate the next definition.

Definition 2.3.2 *Let $A \subseteq \Delta$ be a (probably disconnected) compact set. We define a group homomorphism $\mathbf{H}_A : \pi_1(\Delta - A) \to S^1$ (where S^1 is the unit circle in the complex plane) as follows. Define w as above on the universal covering M of $\Delta - A$. For $[\gamma] \in \pi_1(\Delta - A)$ let $\gamma : [0,1] \to \Delta - A$ be a path representing it and let $\tilde{\gamma}$ be its lift to M. We define*

$$\mathbf{H}_A([\gamma]) = \frac{w(\tilde{\gamma}(1))}{w(\tilde{\gamma}(0))}.$$

The discussion above shows that this is well defined and in S^1. Also if γ goes around the entire set A then $\mathbf{H}_A(\gamma) = 1$. The reason for the letter $\mathbf{H}$ is that this is another case of holonomy. The next lemma relates definitions 2.1.1 and 2.3.2.

Lemma 2.3.3 *If $A \subseteq \Delta$ is compact, ϕ is its potential function and $\mathbf{C}$ is the conductivity, we have*

$$\mathbf{H}_A(\gamma) = \exp(\frac{2\pi i}{\mathbf{C}}\mathbf{H}_\phi(\gamma))$$

for all $\gamma \in \pi_1(\Delta - A)$.

Proof of Lemma 2.3.3:

This is an easy computation using the fact that

$$\phi = B + C \cdot \log |w^n|$$

for appropriate B and C.

$\boxed{2.3.3}$

We can write f as a Laurent series on the region outside the '8' curve; in this way we can write f as a series in φ_n which will converge uniformly on any compact subset of Δ. The coefficient a_n of this series expansion is given by

$$a_n = \frac{1}{2\pi i}\int_\gamma w^{-1-n} f(w)dw = \frac{1}{2\pi i}\int_\gamma w^{-1-n} w' f(z)dz$$

where γ goes once around A. In the case A connected, we proved that if f was small (from the point of view of d_A) then a_n was also small by picking a curve γ close to A. Now we can't do this, and we get a curve of the following kind: γ will go once around A_0, move from A_0 to A_1, go once around A_1 and come back to A_0, possibly in the same way it came. The terms corresponding to going around will be small but there is no reason why the term corresponding to the path should also be, and they can only cancel if $\omega^n = 1$. An explicit counterexample could easily be produced, if necessary. We thus see that with these functions, the obvious generalization of lemma 2.2.8 would be false.

The reader should not think, however, that all this incorrect proof was only a waste of time and effort. Without the knowledge of this failed proof, the correct one would appear unnatural and needlessly complicated. In the above discussion, we also defined φ_n which will play a fundamental role later on. Perhaps most important, we know why and when the old proof fails: this is when w^n is not defined on the complement of A, or when $\omega^n \neq 1$. This still leaves us a small number of cases where w^n *is* defined; in a certain sense, these cases will enable us to prove the entire theorem.

2.3.3 A Preliminary Case

In this subsection we attack the Conductivity Theorem in a simple situation.

Proposition 2.3.4 *Let $A \subseteq \Delta$ be a compact set. Suppose $\mathbf{H}_A(\pi_1(\Delta - A)) \subseteq S^1$ is a finite group. Then*

$$\gamma_{1,2}(A) = \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

Notice to begin with that the hypothesis implies that A has a finite number of connected components. The key observation here is that w^q *is* well defined on $\Delta - A$, as is w^n for any n a multiple of q, where $q = |\mathbf{H}_A(\pi_1(\Delta - A))|$. Very roughly, the proof can be described as follows: for $n \equiv 0 \pmod q$ the old proof works, these values of n give us a space $\mathcal{M}_0$, other spaces $\mathcal{M}_i$ can be defined and compared with $\mathcal{M}_0$, taking the direct sum of these spaces we get $\mathcal{M}$ which gives us the generalized dimension and concludes the proof. Notice the similarity between this proof and that of proposition 2.3.1; these similarities should be kept in mind during the proof. Our first step therefore is to define $\mathcal{M}_0$ and compute its generalized dimension; our second step will be to define $\mathcal{M}_i$, compute their generalized dimensions and prove that indeed $\mathcal{M} = \bigoplus \mathcal{M}_i$.

We now proceed to state and prove a lemma which plays a role similar to that of lemma 2.2.4.

Lemma 2.3.5 *In the situation of proposition 2.3.4 above, define:*

$$\frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A) = \frac{1}{|\log R|}.$$

Define $\varphi_n, n \in \mathbb{N}$ as usual. Let $q = |\mathbf{H}_A(\pi_1(\Delta - A))|$ which is finite by hypothesis. Then the following hold:

(i) Any bounded function $f : \Delta \to \mathbb{C}$ can be written uniquely as a series $f = \sum a_n \varphi_n$. The series converges uniformly and absolutely on any compact subset of Δ.

(ii) $\varphi_0 = 1$ and $\deg(\varphi_n) = n$.

(iii) $|\varphi_n(z)| < K$ for all $z \in \Delta$ and all n, where K is a constant.

(iv) For any $f = \sum a_n \varphi_n \in \mathcal{M}$, $|a_n < 1|$.

(v) For any $R' > R$, there exists a constant $K_{R'}$ such that $d_A(0, \varphi_n) \leq K_{R'} R'^n$ for $n \equiv 0 \pmod{q}$.

(vi) If $d_A(0, f) \leq \epsilon$ then $|a_n| \leq \epsilon K R^{-n}$ for $n \equiv 0 \pmod{q}$, where K is a constant.

Proof of Lemma 2.3.5:

Entirely analogous to proof of lemma 2.2.4, taking into account the observations made in subsection 2.3.2 above.

$\boxed{2.3.5}$

Proof of Proposition 2.3.4:

The first step is so similar to the proof of theorem 2.2.1 that most of the details will be omitted. $\mathcal{M}_0$ is simply defined to be the space of functions generated by φ_n for $n \equiv 0 \pmod{q}$; this can easily be seen to be the space of functions f such that, if $f = \sum a_n w^n$, we have $a_n = 0$ for $n \geq 0$ and $n \not\equiv 0 \pmod q$. After this is said, the proof that

$$\Gamma_{1,2}(\mathcal{M}_0, d_A) = \frac{1}{q}\frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A)$$

follows from the previous lemma just like theorem 2.2.1 followed from lemma 2.2.4.

For $0 \leq i < q$ we define $\mathcal{M}_i$ by induction. We define $\mathcal{M}_{i+1} = \nu \cdot \mathcal{M}_i$ where $\nu = \varphi_1 + K$ where K is a fixed constant such that $\varphi_1 + K$ has no zeroes on A. We have a bijection given by construction between $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$ that takes f to $\nu \cdot f$. This map gives us $\Gamma_{1,2}(\mathcal{M}_i, d_A) = \Gamma_{1,2}(\mathcal{M}_{i+1}, d_A)$ and therefore

$$\Gamma_{1,2}(\mathcal{M}_i, d_A) = \frac{1}{q}\frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A)$$

for all values of i.

Before going on let us introduce some notation. Remember that f can be written as a Laurent series in w (at least in a certain region) and that the non-negative coefficients determine the negative ones. With this fact in mind we can write, for instance,

$$\varphi_1 = w + \cdots$$

where '$\cdots$' stands for the terms in $w^{-1}, w^{-2}, w^{-3}, \ldots$. Using this notation we can write

$$\nu = w + K + \cdots$$

and more generally

$$\nu^n = w^n + Cw^{n-1} + \cdots + Cw + C + \cdots$$

and

$$\nu^n \cdot \varphi_m = w^{n+m} + Cw^{n+m-1} + \cdots + Cw + C + \cdots$$

where the different Cs represent different numbers. This implies that

$$\nu^n \cdot \varphi_m = \varphi_{n+m} + C\varphi_{n+m-1} + \cdots + C\varphi_1 + C.$$

Let us now prove that

$$\bigoplus_{0 \le i < q} \mathcal{M}_i = \mathcal{M}$$

and thus conclude the theorem. This follows from the observations above since we have, in a sense, a triangular matrix. If we have $f = a_n\varphi_n + a_{n-1}\varphi_{n-1} + \cdots + a_0$ we can write $f = a_n\nu^i\varphi_m + f_1$ where $0 \le i < q$, $m \equiv 0 \pmod q$ and $f_1 = a'_{n-1}\varphi_{n-1} + \cdots + a'_0$. The proof proceeds by induction. If f can not be written as a finite sum over φ_n it is still the uniform limit on any compact subset of Δ of such functions. What this means in this situation is that the sequence of approximate decompositions of f into a sum of functions each in $\mathcal{M}_i$ tends to an exact such decomposition.

$\boxed{2.3.4}$

2.3.4 The Final Case

In this subsection we attack the Conductivity Theorem in full generality.

Theorem 2.0.1 *Let $A \subseteq \Delta$ be a compact set. Then*

$$\gamma_{1,2}(A) = \frac{1}{2\pi}\mathbf{C}(\Delta - A; S^1, \partial A).$$

The idea behind this proof is reducing the theorem to the situation of proposition 2.3.4. We do this in two steps. Starting with the set A we consider a harmonic function ϕ_δ

which is close to the potential function ϕ and which has cyclic holonomy outside a small neighbourhood of A. By imitating the proof of proposition 2.3.4 we get an estimate for $\gamma_{1,2}(A)$. After this, we let δ tend to 0 in order to get the exact value of $\gamma_{1,2}(A)$. We divide this proof into lemmas.

Lemma 2.3.6 *Let $A \subseteq \Delta$ be a compact set. Then there exists a harmonic function ϕ_δ defined on $\Delta - A$ satisfying the following properties:*

(i) $z \in S^1 \quad \Rightarrow \quad \phi_\delta(z) = 0.$

(ii) $z \in \partial A \quad \Rightarrow \quad 1 < \phi_\delta(z) < 1 + \delta.$

(iii) Let $A_\delta = A \cup \phi_\delta^{-1}([1, +\infty))$. Then A_δ satisfies the hypothesis of proposition 2.3.4.

Proof of Lemma 2.3.6:

This is done by starting with ϕ_δ equal to $1 + \delta/2$ on ∂A and by making its value slightly larger in some places in such a way that the holonomy group becomes finite.

$\boxed{2.3.6}$

Lemma 2.3.7 *In the situation of the previous lemma we have that, if $\mathbf{C} = \mathbf{C}(\Delta - A; S^1, \partial A)$ and $\mathbf{C}_\delta = \mathbf{C}(\Delta - A_\delta; S^1, \partial A_\delta)$, then*

$$(1 - \delta)\mathbf{C}_\delta < \mathbf{C} < \mathbf{C}_\delta < (1 + \delta)\mathbf{C}.$$

Proof of Lemma 2.3.7:

This is a simple computation.

$\boxed{2.3.7}$

Lemma 2.3.8 *In the situation of lemma 2.3.6 above, define R_δ by*

$$\frac{1}{|\log R_\delta|} = \frac{1}{2\pi}\mathbf{C}(\Delta - A_\delta; S^1, \partial A_\delta).$$

and define R'_δ by

$$\frac{1}{|\log R'_\delta|} = \frac{1}{1+\delta}\frac{1}{2\pi}\mathbf{C}(\Delta - A_\delta; S^1, \partial A_\delta).$$

Define $\varphi_n, n \in \mathbb{N}$ as usual from A_δ, not A. *Let $q = |\mathrm{H}_{A_\delta}(\pi_1(\Delta - A_\delta))|$ which is finite by hypothesis. Then the following hold:*

(i) Any bounded function $f : \Delta \to \mathbb{C}$ can be written uniquely as a series $f = \sum a_n\varphi_n$. The series converges uniformly and absolutely on any compact subset of Δ.

(ii) $\varphi_0 = 1$ and $\deg(\varphi_n) = n$.

(iii) $|\varphi_n(z)| < K$ for all $z \in \Delta$ and all n, where K is a constant.

(iv) For any $f = \sum a_n\varphi_n \in \mathcal{M}$, $|a_n < 1|$.

(v) For any $R' > R_\delta$, there exists a constant $K_{R'}$ such that $d_A(0, \varphi_n) \leq K_{R'}R'^n$ for $n \equiv 0 \pmod q$.

(vi) If $d_A(0, f) \leq \epsilon$ then $|a_n| \leq \epsilon K {R'_\delta}^{-n}$ for $n \equiv 0 \pmod q$, where K is a constant.

Proof of Lemma 2.3.8:

Entirely analogous to proof of lemma 2.2.4, taking into account the observations made in subsection 2.3.2 above.

$\boxed{2.3.8}$

Lemma 2.3.9 *Let $A \subseteq \Delta$ be a compact set. For any $\delta > 0$, define R_δ and R'_δ as in lemma 2.3.8. Then we have*

$$\frac{1}{|\log R'_\delta|} < \gamma_{1,2}(A) < \frac{1}{|\log R_\delta|}.$$

Proof of Lemma 2.3.9:

This is analogous to the proof of theorem 2.2.1 or proposition 2.3.4 with lemma 2.3.8 above playing the role of lemma 2.2.4 or lemma 2.3.5. More exactly, we get

$$\gamma_{1,2}(A) > \frac{1}{|\log R'_\delta|}$$

by a straightforward generalization of lemma 2.2.5, in other words, by building a large packing set and we get

$$\gamma_{1,2}(A) < \frac{1}{|\log R_\delta|}$$

by a corresponding generalization of lemma 2.2.6, i.e., by building a small covering set.

$\boxed{2.3.9}$

Proof of Theorem 2.0.1:

This is an immediate corollary of lemma 2.3.9.

$\boxed{2.0.1}$

Chapter 3

Further Results

3.0 Introduction

In this chapter we prove several other minor facts about spaces of functions. In the next section we study $\gamma_{2,1}(A)$ and $\gamma_{1,r}(A)$ in the cases where A is so small that $\gamma_{1,2}(A) = 0$. In another section we study different spaces and slightly different metrics in these spaces in order to see how the results from the last two chapters would be affected.

3.1 Small Sets

3.1.1 Countable Sets

We already saw that if A is infinite, we have $\gamma_{1,1}(A) = +\infty$. It is an immediate consequence of the Conductivity Theorem that if $A \subseteq \Delta$ is a countable compact set then $\gamma_{1,2}(A) = 0$. We shall now see that these are about the only restrictions on the "size" of a general countable set. (From now on in this subsection 'countable' will stand for 'infinite countable'.)

Example 3.1.1 *There is a compact countable set A, $A \subseteq \Delta$, A of type $\omega+1$, with $\gamma_{2,1}(A) =$ 2.*

(A set of type $\omega + 1$ is a compact set with exactly one accumulation point.)

A shall be the image of a sequence (a_n) to be constructed satisfying $\lim a_n = 0$ plus the point 0. Intuitively, the faster (a_n) tends to 0, the smaller $\gamma_{2,1}(A)$ shall be; therefore, we want a sequence (a_n) that tends to 0 rather slowly. Let us consider two auxiliary sequences (R_i) and (ϵ_i) to be specified later, both tending to 0. Now let us pick a finite number of points on the circle of radius R_i around the origin a distance at most ϵ_i apart from their neighbours. Let now the sequence (a_n) start with the previously selected points on the circle of radius R_0 and list them, then do the same thing on the circle of radius R_1 and so on. Let us prove that by picking the right (R_i) and (ϵ_i) we can force $\gamma_{2,1}(A) = 2$.

Take (for instance) $R_i = (1/2)^{(i+2)}$. Let B be an $(\epsilon + 2\epsilon_i)$-packing set for $(\mathcal{M}, d_{S^1_{R_i}})$ where $S^1_{R_i}$ is the circle of radius R_i around the origin. We can see from the Schwartz-Pick theorem that B is an ϵ-packing set for $(\mathcal{M}, d_A)$. We know that

$$\lim_{\epsilon \to 0} \frac{\log \log NP_\epsilon(\mathcal{M}, d_{S^1_{R_i}})}{\log |\log \epsilon|} = 2$$

From this it can easily be seen that we can select a value for ϵ_i which will imply

$$\frac{\log \log NP_{\epsilon_i}(\mathcal{M}, d_A)}{\log |\log \epsilon_i|} \geq 2 - (1/2)^{(i+2)}$$

With such a selection of ϵ_i we have

$$\gamma_{2,1}(A) = \lim_{\epsilon \to 0} \frac{\log \log NP_\epsilon(\mathcal{M}, d_A)}{\log |\log \epsilon|} \geq 2$$

and, since we already knew that $\gamma_{2,1}(A) \leq 2$ we have $\gamma_{2,1}(A) = 2$. This concludes the example.

$\boxed{3.1.1}$

This construction can easily be generalized in order to get other countable sets A with $\gamma_{2,1}(A) = r$ for any $1 \leq r \leq 2$.

3.2 Other Spaces, Other Metrics

3.2.1 New Metrics

Definition 3.2.1 *Let $A \subseteq \Delta$, $A \neq \emptyset$, and $f_1, f_2 \in \mathcal{H}$. For $n \geq 0$, we define*

$$d_A^{(n)}(f_1, f_2) = \sup_{z_0 \in A,\ i \leq n} |a_i - b_i|$$

where

$$f_1(w) = a_0 + a_1 w + a_2 w^2 + a_3 w^3 + \cdots$$

$$f_2(w) = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots$$

and

$$w = \frac{z - z_0}{z\overline{z_0} - 1}.$$

In particular we have $d_A^{(0)} = d_A$. We list some trivial consequences of the definition as a proposition.

Proposition 3.2.2 *The following are properties of $d_A^{(n)}$.*

(i) For any $A \subseteq \Delta$ and any $n \geq 0$, $d_A^{(n)}$ is a pseudometric on M with $0 \leq d_A^{(n)} \leq 2$.

(ii) If $A \subseteq B$, $d_A^{(n)} \leq d_B^{(n)}$.

(iii) $d_A^{(n)} = d_{\bar{A}}^{(n)} = d_{\partial A}^{(n)}$.

(iv) If $A \subseteq \Delta$ has an accumulation point in Δ, $d_A^{(n)}$ is a metric.

(v) If φ is a conformal bijection from Δ to itself, it induces an isomorphism between the two metric spaces $(\mathcal{M}, d_A^{(n)})$ and $(\mathcal{M}, d_{\varphi(A)}^{(n)})$.

(vi) If $m \leq n$, $d_A^{(m)} \leq d_A^{(n)}$.

(vii) If $A \subseteq D$ where D is the disc of radius R around the origin then

$$d_A^{(n)}(f_1, f_2) \leq \max_{i \leq n} \frac{1}{i!} d_A(f_1^{(i)}, f_2^{(i)}).$$

Proof of Proposition 3.2.2:

This is entirely analogous to proposition 1.2.2 except for items *(vi)* and *(vii)*, which are straightforward.

$\boxed{3.2.2}$

3.2.2 Hyperdimension and Generalized Dimension

Definition 3.2.3 *If $A \subseteq \Delta$, we define*

$$\gamma^{(n)+}_{m,r}(A) = \Gamma^{+}_{m,r}(\mathcal{M}, d^{(n)}_A), \qquad \gamma^{(n)-}_{m,r}(A) = \Gamma^{-}_{m,r}(\mathcal{M}, d^{(n)}_A),$$

We drop the + and − signs when they are superfluous, i.e., when the corresponding limit exists.

In particular, $\gamma^{(0)}_{m,r} = \gamma_{m,r}$. The following conjecture is an obvious generalization of conjecture 1.2.6.

Conjecture 3.2.4 *$\gamma^{(n)}_{m,r}(A)$ are always defined.*

We shall see that this is almost equivalent to the original conjecture.

Next we state some easy consequences of proposition 3.2.2 and definition 3.2.3.

Proposition 3.2.5

(i) For any $n \geq 0$,

$$\gamma^{(n)}_{2,1}(\{0\}) = 1, \qquad \gamma^{(n)}_{1,1}(\{0\}) = 2(n+1).$$

(ii) If $A \subseteq B$ then (for any r)

$$\gamma^{(n)-}_{m,r}(A) \leq \gamma^{(n)-}_{m,r}(B), \qquad \gamma^{(n)+}_{m,r}(A) \leq \gamma^{(n)+}_{m,r}(B).$$

(iii)

$$\gamma^{(n)-}_{m,r}(A) = \gamma^{(n)-}_{m,r}(\bar{A}) = \gamma^{(n)-}_{m,r}(\partial A),$$

$$\gamma^{(n)+}_{m,r}(A) = \gamma^{(n)+}_{m,r}(\bar{A}) = \gamma^{(n)+}_{m,r}(\partial A).$$

(iv)

$$\gamma_{2,1}^{(n)+}(A \cup B) \le \max(\gamma_{2,1}^{(n)+}(A); \gamma_{2,1}^{(n)+}(B));$$

$$\gamma_{1,r}^{(n)+}(A \cup B) \le \gamma_{1,r}^{(n)+}(A) + \gamma_{1,r}^{(n)+}(B).$$

(v) If $A_i, 0 \le i < n$ is a family of isometric sets (with respect to the Poincaré Metric) then

$$\gamma_{2,1}^{(n)-}(\bigcup A_i) = \gamma_{2,1}^{(n)-}(A_0),$$

$$\gamma_{1,r}^{(n)-}(\bigcup A_i) \le n\gamma_{1,r}^{(n)-}(A_0).$$

(vi) If A is a finite set,

$$\gamma_{2,1}^{(n)}(A) = 1, \qquad \gamma_{1,1}^{(n)}(A) = 2(n+1)|A|.$$

(vii) If $n_0 \le n_1$ then

$$\gamma_{m,r}^{(n_0)-}(A) \le \gamma_{m,r}^{(n_1)-}(A), \qquad \gamma_{m,r}^{(n_0)+}(A) \le \gamma_{m,r}^{(n_1)+}(A).$$

Proof of Proposition 3.2.5:

(i) is follows from the fact that $(\mathcal{M}, d_{\{0\}})$ is essentially isomorphic to $(\Delta, ||)$, meaning it is isomorphic once you identify points which are at zero distance. The disc is known to have Hausdorff dimension 2. The case $n > 0$ is similar.

(ii) and *(iii)* are obvious consequences of the corresponding items of proposition 1.2.2.

From $d_{A\cup B} = \max(d_A, d_B)$ it follows that $(\mathcal{M}, d_{A\cup B})$ is a subspace of $(\mathcal{M}, d_A) \times (\mathcal{M}, d_B)$, and therefore has smaller $\Gamma_{2,1}^{+}$ and $\Gamma_{1,r}^{+}$. This observation together with proposition 1.1.8 gives us *(iv)*. *(v)* follows similarly from proposition 1.1.8 since now the spaces $(\mathcal{M}, d_{A_i})$ are all isomorphic.

(vi) is proved by induction from *(i)* and *(iv)*.

(vii) follows directly from proposition 1.2.2.

$\boxed{3.2.5}$

3.2.3 A Theorem

We defined all these more general metrics because of their geometric meaning. The following theorem tells us that these superscripts do not make a very big difference after all.

Theorem 3.2.6 *For any $A \subseteq \Delta$, for any $n \geq 0$*

$$\gamma_{2,1}^{(n)+}(A) = \gamma_{2,1}^{+}(A), \qquad \gamma_{2,1}^{(n)-}(A) = \gamma_{2,1}^{-}(A),$$

and, for any $r > 1$, if A has a finite number of connected components,

$$\gamma_{1,r}^{(n)+}(A) = \gamma_{1,r}^{+}(A), \qquad \gamma_{1,r}^{(n)-}(A) = \gamma_{1,r}^{-}(A).$$

Proof of Theorem 3.2.6:

We begin by observing that for $\gamma_{2,1}$ the theorem is trivial, since $(\mathcal{M}, d_A^{(n)})$ is equivalent to $(\mathcal{M}, d_A^{(n)})$ which is in turn isomorphic to a subspace of $(\mathcal{M}, d_A)^n$.

We now prove the theorem in the case where A has a finite number of connected components. We do this by observing that $(\mathcal{M}, d_A^{(n)})$ is isomorphic to a subspace of $(\mathcal{M}, d_A) \times (\Delta, ||)^k$ where k is the number of connected components.

$\boxed{3.2.6}$

3.2.4 Other Metrics

We shall now proceed to study other metrics on $\mathcal{M}$, similar to $d_A^{(n)}$ under many aspects. These should be seen as variants of $d_A^{(n)}$, rather than as completely new metrics. We shall prove that the study of $\Gamma_{2,1}$ of $\Gamma_{1,r}$ for these metrics is superfluous, since it reduces to that of $d_A^{(n)}$. The reason for introducing these new metrics is that in many cases they are more intuitive, more meaningful, or easier to use than $d_A^{(n)}$.

Definition 3.2.7 *For $A \subseteq \Delta$ a non-empty compact set, we define*

$$\mathrm{d}_A^{(n)}(f_0, f_1) = \max_{0 \le i \le n} d_A(f_0^{(i)}, f_1^{(i)}).$$

Proposition 3.2.8 *For any non-empty compact set $A \subseteq \Delta$, $\mathrm{d}_A^{(n)}$ and $d_A^{(n)}$ are equivalent (in the sense of definition 1.1.6).*

Proof of Proposition 3.2.8:

This is a simple computation.

$\boxed{3.2.8}$

Definition 3.2.9 *For $A \subseteq \Delta$ a non-empty compact set, we define*

$$\delta_A(f_0, f_1) = \sup_{z \in A} d_h(f_0(z), f_1(z))$$

where d_h denotes the hyperbolic metric on Δ. We define

$$\delta_A^R(f_0, f_1) = \delta_A(Rf_0, Rf_1) = \sup_{z \in A} d_h(Rf_0(z), Rf_1(z))$$

where $0 < R < 1$ is a constant.

Proposition 3.2.10 *For any non-empty compact set $A \subseteq \Delta$ and for any $0 < R < 1$, δ_A^R and d_A are equivalent (in the sense of definition 1.1.6).*

Proof of Proposition 3.2.10:

This is a simple computation.

$\boxed{3.2.10}$

3.2.5 Other Spaces

In this subsection we shall briefly consider some other spaces of functions.

One natural space to consider is the space $\mathcal{S}$ of meromorphic functions defined on the disc. The most natural way of extending the definition of d_A to this situation is to use the metric on the sphere given by its standard immersion in $\mathbb{R}^3$. Calling this metric d, we have:

Definition 3.2.11 *Let $A \subseteq \Delta$ and let $f_0, f_1 \in \mathcal{S}$ be arbitrary. Then*

$$d_A^{\mathcal{S}}(f_0, f_1) = \sup_{z \in A} d(f_0(z), f_1(z)).$$

This space is not at all fit for the kind of questions we want to ask as the next proposition shows.

Proposition 3.2.12 *For any $A \subseteq \Delta$, A with non-empty interior, for any $\epsilon < 2$,*

$$NP_\epsilon(\mathcal{S}, d_A^{\mathcal{S}}) = +\infty.$$

This is in fact true for any infinite A.

We prove this proposition with the aid of the next lemma.

Lemma 3.2.13 *For any $f \in \mathcal{S}$, for any (small) open subset B of Δ there exists $g \in \mathcal{S}$ with $d^{\mathcal{S}}_{\Delta-B}(f,g) < \epsilon$, $d^{\mathcal{S}}_{B}(f,g) = 2$.*

Proof of Lemma 3.2.13:

We take $g = fh$ with $h = 1 + k\frac{1}{z-z_0}$ with $z_0 \in B$ and k a small constant.

$\boxed{3.2.13}$

Proof of Proposition 3.2.12:

Just take an infinite family (B_i) of disjoint open subsets of A and use the above lemma.

$\boxed{3.2.12}$

Another space we can consider is the space of Lipschitz functions with constant 1 from one compact space to another compact space. One reason for considering such a space is that the Schwartz-Pick Theorem tells us that the functions in $\mathcal{M}$ are Lipschitz with constant 1 if Δ is given the Poincaré metric.

Let K_0 and K_1 be two compact spaces; let $\mathcal{L}$ be the space of Lipschitz functions with constant 1 from K_0 to K_1; give $\mathcal{L}$ the supremum norm. $\mathcal{L}$ is also compact and $NP_\epsilon(\mathcal{L})$ and $NC_\epsilon(\mathcal{L})$ will be finite. We can therefore talk about $\Gamma_{n,r}(\mathcal{L})$ but the next proposition shows there is usually not much to talk about.

Proposition 3.2.14 *Let $K_0 = K_1 = [0,1]$; $\Gamma_{n,r}(\mathcal{L}) = +\infty$ for any n, r.*

Proof of Proposition 3.2.14:

All we have to do is construct a large packing set. Let A be a maximal (2ϵ)-packing set of $[0,1]$ and let $n = |A|$; we have $n > 1/2\epsilon$. We can now select the value of the function to be either $(1/2)+\epsilon$ or $(1/2)-\epsilon$ on each point of A, the selection on different points being independent. This gives us an ϵ-packing set for $\mathcal{L}$ with at least $2^{1/2\epsilon}$ elements. In other words, $NC_\epsilon \geq 2^{1/2\epsilon}$. This gives us the desired result.

$\boxed{3.2.14}$

The reader probably realizes that the packing set we constructed wasted most of $\mathcal{L}$. This reflects the fact that $\mathcal{L}$ is much larger than spaces with finite $\Gamma_{n,r}$, even for large values of n and r.

Chapter 4

More General Metrics

4.0 Introduction

In this chapter we consider generalizations of d_A which, while they may seem artificial when defined, will prove to be important for the applications. We state what we believe to be the correct genearlization of the Conductivity Theorem as a conjecture; the fact that we can not prove it at this time accounts for the brevity of this chapter. We prove some weaker results which allow us to work out some of the applications.

4.1 Introducing the New Metrics

4.1.1 Generalizing d_A

If $f : [0, +\infty) \to [0, +\infty)$ takes 0 to 0, is increasing and has the concavity always down then for any metric d, $f \circ d$ is also a metric. The concavity condition is necessary in order to guarantee that the triangle inequality will still hold. In particular, we observe that if d is a metric then so is d^r if $0 < r \leq 1$. For $r > 1$, however, we usually do not get a metric since the triangle inequality will probably fail.

Let us take look at how these metrics compare to d and one another. For values of d less than one, the smaller r is, the larger d^r will be. If our metric space is bounded, we see that if we make r smaller, d^r will become essentially larger in the sense that

$$r_1 < r_2 \qquad \Longrightarrow \qquad (\exists C) d^{r_1} > C d^{r_2}.$$

We see that things are simpler if the distances are bounded above by 1 from the start; we also see that taking $r = 1/p$ will mean that for larger p we have a larger metric.

We can use the ideas above to introduce the following definition.

Definition 4.1.1 *For $p : \Delta \to [0, +\infty]$ we define*

$$d_p(f, g) = 2 \sup_{z \in A} \left(\frac{|f(z) - g(z)|}{2} \right)^{1/p(z)}$$

where in the case $p(z) = 0$ we adopt the convention that

$$\left(\frac{|f(z) - g(z)|}{2} \right)^{1/p(z)} = 0$$

and in the case $p(z) = +\infty$ we adopt the convention that

$$\left(\frac{|f(z) - g(z)|}{2}\right)^{1/p(z)} = 1$$

unless $|f(z) - g(z)| = 0$ in which case the expression above is said to have value 0.

Intuitively, p gives a measure of how much a point counts: the larger $p(z)$ is, the more sensitive d_p is to the value of f at z. Notice that $d_p(f, g)$ is always defined and that we always have $0 \leq d_p(f, g) \leq 2$. It is not at all clear, however, just when d_p will be a metric. We proceed now to study this question and to study d_p. Let us observe right now, however, that if χ_A is the characteristic function of a set A we have $d_{\chi_A} = d_A$. We can therefore think of d_p as a generalization of d_A.

4.1.2 On d_p

We could now study d_p as we did d_A. The analogies are so strong and so obvious, however, that we shall be much more concise.

Lemma 4.1.2 *For all p and all $f, g \in \mathcal{M}$, $0 \leq d_p(f, g) \leq 2$ and $d_p(f, f) = 0$. If* supp(p) *has an accumulation point in Δ, then $d_p(f, g) = 0$ implies $f = g$.*

Proof of Lemma 4.1.2:

We already saw why the first part is true. The second part follows from the analiticity of f and g and analytic continuation.

$\boxed{4.1.2}$

Lemma 4.1.3 *If p never assumes values in $(0,1)$ and $\operatorname{supp}(p)$ has an accumulation point in Δ then d_p is a metric.*

It is important to note that the opposite implication is *not* true; we shall later consider many cases where p does assume values between 0 and 1 but d_p is still a metric.

Proof of Lemma 4.1.3:

The only non-trivial part is the triangle inequality and this follows from our restrictions to the values which p can assume.

$\boxed{4.1.3}$

Lemma 4.1.4 *If $p_1(z) \leq p_2(z)$ for all $z \in \Delta$, then $d_{p_1} \leq d_{p_2}$.*

Proof of Lemma 4.1.4:

Easy.

$\boxed{4.1.4}$

Lemma 4.1.5 *If p is bounded and has compact support and d_p is a metric then $(\mathcal{M}, d_p)$ is a compact metric space.*

Proof of Lemma 4.1.5:

Just notice that the conditions imply $p \leq K \cdot \chi_A$ for a compact set A and that $(\mathcal{M}, d_{K \cdot \chi_A})$ has the same topology as $(\mathcal{M}, d_A)$.

$\boxed{4.1.5}$

Proposition 4.1.6 *If ϕ_p is the smallest superharmonic function such that $\phi_p(z) \geq p(z)$ for all z then $d_{\phi_p} = d_p$.*

Proof of Proposition 4.1.6:

All we have to show is that if $-\log(|f(z)|) \geq K \cdot p(z)$ for all z then also $-\log(|f(z)|) \geq K \cdot \phi_p(z)$ for all z, but this follows from the fact that $-\log(|f(z)|)$ is superharmonic.

$\boxed{4.1.6}$

4.2 Towards a Generalization of the Conductivity Theorem

4.2.1 The Generalized Conductivity Theorem

The next definition is the natural generalization of $\gamma_{n,r}(A)$.

Definition 4.2.1 *When p is such that d_p is a metric and $(\mathcal{M}, d_p)$ is compact, we define $\gamma_{n,r}(p) = \Gamma_{n,r}((\mathcal{M}, d_p))$.*

Many results about $\gamma_{n,r}(A)$ generalize easily to $\gamma_{n,r}(p)$. For instance, just as $\gamma_{2,1}(A) = 2$ for all but very small A so $\gamma_{2,1}(p) = 2$ for all but very small p. Since the Conductivity Theorem was our main result about $\gamma_{n,r}(A)$, we would naturally expect its generalization to be a most important result about $\gamma_{n,r}(p)$. We now state as a conjecture what we have strong reason to believe to be the correct generalization of the Conductivity Theorem to the new situation.

Conjecture 4.2.2 *Let p be as in the previous subsection. Let ϕ_p be the smallest superharmonic function satisfying $\phi_p \geq p$. Then*

$$\gamma_{1,2}(p) = \frac{1}{\pi}\mathbf{E}(\phi_p).$$

We shall not prove this conjecture in this work. In the next subsection we prove some easy inequalities which will serve as poor substitutes for a proof of the conjecture. Notice that the Conductivity Theorem is a special case of this conjecture.

4.2.2 Some Estimates

Proposition 4.2.3 *Let p assume value r in A and 0 elsewhere. Then*

$$\gamma_{1,2}(p) = r^2\gamma_{1,2}(A).$$

Proof of Proposition 4.2.3:

Easy.

$\boxed{4.2.3}$

Proposition 4.2.4 *Let p be as in the previous section. Let ϕ_p be the smallest superharmonic function satisfying $\phi_p \geq p$. Let A_r be the inverse image of r under ϕ_p. Then*

$$\gamma_{1,2}(p) \geq r^2\gamma_{1,2}(A_r).$$

Proof of Proposition 4.2.4:

This is an easy consequence of the previous proposition.

$\boxed{4.2.4}$

Chapter 5

Specifying Functions

5.0 Introduction

In this chapter we put the ideas of the previous chapters into use: we study the question of how most efficiently to specify holomorphic functions. We shall give an explicit tentative answer and show that no method can be much better than ours in a certain precise sense. We then study analytic continuation. We show that, although theoretically possible, analytic continuation is impractical in all but the simplest situations. The precision with which the function needs to be known at the original point or region in order to give a decent approximation at the final point or region may be such as to render the problem physically impossible. All of these statements shall be made precise on occasion. After this part of our work had been done, we found out about the work of Vituškin (see [6]) who addresses this question of specifying functions in many different situations and derives many interesting results.

5.1 Specifying Functions

5.1.1 Statement of the Problem

In this section we want to consider the question of how best to specify functions. We interpret a method of specifying a function as a pair of maps *enc* and *dec* as follows:

$$\begin{array}{rccc} enc: & \mathcal{M} & \to & \mathcal{C} \\ & f & \mapsto & enc(f) \end{array} \qquad \begin{array}{rccc} dec: & \mathcal{C} & \to & \mathcal{M} \\ & n & \mapsto & dec(n) \end{array}$$

where $\mathcal{C}$ is a finite set. We shall call $|\mathcal{C}|$ the *size* of the method. Intuitively, $\mathcal{C}$ is the space of names of functions under a certain specification method. If f is a function, $enc(f)$ is "the name of the function". Also, if n is a name, $dec(n)$ is "the function named by n". (The names *enc* and *dec* stand for "encode" and "decode".) The expressions above are inside quotes because it is, of course, impossible to find a bijection between $\mathcal{M}$ and $\mathcal{C}$ since one of these sets is finite and the other uncountable. The impossibility of having a perfect specification method leads us to ask how good a specification method we can find.

This is meaningless, however, until we define what we mean by a 'good' specification method and say how we can compare the quality of different methods. The simplest possibility is to take the greatest possible error you can get by encoding and decoding a function to be a measure of the 'badness' of the method. The error mentioned above is, of course,

$d_A(f, dec(enc(f)))$. We shall call this the *precision* of the method. The smaller the precision the better the method.

Proposition 5.1.1 *If N is the size of a specification method and ϵ is its precision we have*

$$N \geq NC_\epsilon(\mathcal{M}, d_A).$$

Proof of Proposition 5.1.1:

It is enough to observe that $dec(enc(\mathcal{M}))$ is an ϵ-covering set of size N.

$\boxed{5.1.1}$

We define a *specification procedure* to be a family of specification methods, one for each size. We define the precision of a specification procedure to be the function p which takes each N to the precision of the method of size N. The faster p tends to zero, the better the procedure.

Proposition 5.1.2 *For any specification procedure we have*

$$\lim_{N\to\infty} \frac{\log^2(p(N))}{\log(N)} \leq \frac{1}{\gamma_{1,2}(A)}$$

Proof of Proposition 5.1.2:

This is a simple consequence of the previous proposition and the definition of $\gamma_{1,2}$.

$\boxed{5.1.2}$

We did all of this for d_A only but everything, including the proofs, carries over trivially to the situation d_p. There is of course, the difference that we do not *know* the value of $\gamma_{1,2}(p)$: we only have a conjecture.

Notice that the expression inside the limit is a measure of the speed with which $p(N)$ tends to zero as N tends to infinity. The proposition above puts a bound on the quality of any procedure.

Let us try to interpret the expression above. $\log(N)$ is the number of bits of information needed in order to store or communicate the name of a function up to a multiplicative constant. $\log(p(N))$ is the number of decimal places of the value of the function which are actually known up to another multiplicative constant. The inequality above is telling us that if we are to be given n bits of information about the function we can hope to know only the first m decimal places of the value of the function on A where

$$m = \sqrt{\frac{Kn}{\gamma_{1,2}(A)}},$$

K being a constant.

5.1.2 Explicit Procedures

In the previous subsection we saw bounds on how good a specification procedure can ever hope to be. We saw, however, no explicit examples. We proceed now to consider a few concrete and explicit procedures and to study how good they are. We shall eventually meet an example of a procedure that actually realizes the equality in proposition 5.1.2.

Example 5.1.3 *We can specify a function by giving its power series.*

This is one of the most obvious ways of specifying a function. One thing that we have to worry about is with what precision we are going to specify each coefficient. We can easily see that this precision should not be the same for all coefficients; precision should start high and go down as we look at higher coefficients until we get to the point where we need no precision and therefore forget about the coefficient completely. All that is said below assumes we handled this problem correctly.

The power series method is not bad if your domain is a round ball and you take the series at the center; in this case we do realize the equality in proposition 5.1.2. In other situations using this method is equivalent to ignoring the geometry of A and pretending that it is a round ball when it is in fact not. It is easy to see that doing this would give us

$$\lim_{N\to\infty} \frac{\log^2(p(N))}{\log(N)} = \frac{1}{\gamma_{1,2}(B)}$$

where B is a ball containing A.

Example 5.1.4 *We can specify a function by giving its value at sample points.*

Here the main question is: what sample points should we take? We then have to address the other question which is: with what precision should the value of the function be given? In any case, we can intuitively see that this method always involves some waste of information: if two points are close and we know the value of a function at one of them then we know the approximate value of the same function at the other point. All of this shows that this method is hard to use optimally and even if so used is unlikely to be very good. We shall leave the question of exactly how good this method can be at its best unanswered, but we know it can never be *much* better than our next method.

Example 5.1.5 *We can specify a function by giving its series expansion in terms of* φ_n.

In the case of A connected, this is the 'good' method we promised: if we specify the coefficients of this series expansion with the right precision we can realize the equality in proposition 5.1.2. The proof of this is in fact contained in the proof of the Conductivity Theorem which also describes exactly what precision we should use for each coefficient.

In the case of A disconnected we have to change this series expansion as in the proof of the corresponding case of the Conductivity Theorem.

Chapter 6

On Analytic Continuation

6.0 Introduction

In this chapter we finally arrive at the problem of the practicality of analytical continuation. This is the problem that gives the title to this work and it is also our original motivation. We show that, although theoretically possible, analytic continuation is impractical in all but the simplest situations. The precision with which the function needs to be known at the original point or region in order to give a decent approximation at the final point or region may be such as to render the problem physically impossible. All of these rather vague statements are defined precisely, often in more than one way. Not all questions are answered, however, and several interesting conjectures are left to be answered in the future.

6.1 A Simple Interpretation

6.1.1 A Simple Interpretation

Consider that we know a function on a set A and wish to find out its value at a point z, the interesting case being $z \notin A$. We could do this by analytic continuation on a path from A to z. This shows that finding $f(z)$ is theoretically possible. Let us investigate the question of how hard it is to do this "in practice". This question is of course subject to more than one interpretation; we shall consider a few of them in this chapter.

One way of interpreting the assumption that the function is known on A is to say that we know f as a point in the metric space $(\mathcal{M}, d_A)$. Following this point of view, we consider $f(z)$ as a function on the space $(\mathcal{M}, d_A)$, in other words, as a function of f. We shall call this function $\mathbf{z}$ so that we have $\mathbf{z}(f) = f(z)$. Provided A is infinite, we know that $\mathbf{z}$ is uniformly continuous. For each ϵ we define $\delta(\epsilon)$ to be the largest positive number such that $d_A(f, g) < \delta(\epsilon)$ implies $|\mathbf{z}(f) - \mathbf{z}(g)| < \epsilon$. We now see that the function δ gives a possible answer to our question: the faster it tends to zero, the harder it is to do analytic continuation. We proceed now to study the function δ.

Before doing so, however, let us introduce some notation. Recall that when $A \subseteq \Delta$ is compact we can define a unique harmonic function ϕ on $\Delta - A$ such that $\phi(z)$ tends to 0 as z tends to $\partial\Delta$ but $\phi(z)$ tends to 1 as z tends to ∂A. We call such function the *potential function* for A and denote it by ϕ_A so that its value on z shall be denoted by $\phi_A(z)$.

Proposition 6.1.1 *Let A and z be as above. Let $\delta(\epsilon)$ be defined as above. Then we have*

$$\lim_{\epsilon\to 0}\frac{\log\delta}{\log\epsilon}=\frac{1}{\phi_A(z)}.$$

The proposition above tells us that z is Hölder and tells us with what constant.

Proof of Proposition 6.1.1:

Consider the harmonic function $\psi(z)=-\log(|f(z)|)$ on $\Delta - A$. This function satisfies $\psi(z)\geq 0$ for $z\in S^1$ and $\psi(z)\geq -\log(d_A(f,0))$ for $z\in\partial A$; also if z_0 is a discontinuity of $\psi(z)$ we have $\lim_{z\to z_0}\psi(z)=+\infty$. All of this implies that $\psi(z)\geq -\log(d_A(f,0))\phi_A(z)$. In other words, $|\log(f(z))|\geq|\log(d_A(f,0))|\phi_A(z)$ which gives us

$$\lim_{\epsilon\to 0}\frac{\log\delta}{\log\epsilon}\leq\frac{1}{\phi_A(z)}.$$

The opposite inequality is guaranteed by $\varphi_n(z)$, the functions constructed in the proof of the Conductivity Theorem, where in the case of A not connected n has to be selected to be a multiple of a certain p again just as in the proof of the Conductivity Theorem.

$\boxed{6.1.1}$

How would we extend this result from the case of d_A to the case of d_p? The following is the natural generalization. We shall be able to prove it and we do *not* need the more general version of the Conductivity Theorem which we conjectured to be true.

Proposition 6.1.2 *Let p and z be as above with the obvious generalizations. Let $\delta(\epsilon)$ be defined as above. Then we have*

$$\lim_{\epsilon\to 0}\frac{\log\delta}{\log\epsilon}=\frac{1}{\phi_p(z)}.$$

Remember that ϕ_p is the smallest superharmonic function larger than p.

Proof of Proposition 6.1.2:

We think of this proposition as two inequalities. One of them is entirely analogous to the first half of the proof of proposition 6.1.1 above. Notice that we have $\psi(z) \geq -\log(d_A(f,0))\phi_p(z)$ where ϕ_p is *not* harmonic, which implies the same inequality everywhere. The other direction follows from proposition 2.1.13.

$\boxed{6.1.2}$

We could, of course, have given this second proof in general, but we thought it would be good to have a more concrete proof of a simple case.

6.2 The Information Approach

6.2.1 Asking Questions

In the previous section we considered the following question: "If we want $f(z)$ with a certain precision, with what precision do we need to know f on A?". This is a valid question, but it is not at all clear that it measures the difficulty of doing analytic continuation. After all, knowing f with a certain precision on a small set is easy, but knowing it with the same precision on a large set is hard. The following question corresponds more closely to the spirit of the original one: "How many bits of information about f do we need in order to deduce $f(z)$ with a certain precision?". We clearly have to say what kind of questions we have in mind or we could just forget about A and ask directly about $f(z)$. Let us explore several possibilities. For all possibilities considered, we shall define $N_\epsilon(A,z)$ to be the number of questions concerning $f(w)$ for $w \in A$ necessary in order to find out the value of $f(z)$ within an error of ϵ. We are interested in estimating the size of $N_\epsilon(A,z)$.

The most restrictive possibility is to say that you have to ask questions that will tell you the value of $f(z)$ for $z \in A$ within an error of δ and then reduce the problem to that considered in the previous subsection. What we saw previously tells us that, with this very restrictive approach, we have

$$\lim_{\epsilon \to 0} \frac{N_\epsilon(A,z)}{\log^2 \epsilon} = \frac{1}{\log 2} \frac{\gamma_{2,1}(A)}{\phi_A^2(z)}.$$

Notice that $\mathrm{C}\phi^{-2}$ need not always increase when A becomes larger; this shows that this possibility is *too* restrictive. We can consider a less restrictive possibility as follows: you have first to find out the value of $f(z)$ for $z \in A' \subseteq A$ where A' will be any subset of A which you are free to choose. With this more plausible approach we have

$$\lim_{\epsilon \to 0} \frac{N_\epsilon(A,z)}{\log^2 \epsilon} = \min_{A' \subseteq A} \frac{1}{\log 2} \frac{\gamma_{2,1}(A')}{\phi_{A'}^2(z)}.$$

It is now natural to say that the most general, the 'right' answer, corresponds to paying more attention to some parts of A than to others but that we can divide our attention in any way we want. This corresponds to taking any p provided only $\mathrm{supp}(p) \subseteq A$. This amounts to saying that

$$\lim_{\epsilon \to 0} \frac{N_\epsilon(A,z)}{\log^2 \epsilon} = \min_{\mathrm{supp}(p) \subseteq A} \frac{1}{\log 2} \frac{\gamma_{2,1}(p)}{\phi_p^2(z)}.$$

Our conjectures would then give us

$$\lim_{\epsilon \to 0} \frac{N_\epsilon(A,z)}{\log^2 \epsilon} = \min_{\mathrm{supp}(p) \subseteq A} \frac{1}{\pi \log 2} \frac{\mathrm{E}(\phi_p)}{\phi_p^2(z)}.$$

Let us restate this conclusion as a conjecture.

Conjecture 6.2.1 *Suppose a function $f \in \mathcal{M}$ is known in a compact set A. Suppose we want to find out its value in z. In order to determine $f(z)$ within an error of ϵ we need to ask N_ϵ questions about f where*

$$\lim_{\epsilon \to 0} \frac{N_\epsilon(A,z)}{\log^2 \epsilon} = \min_{\operatorname{supp}(p) \subseteq A} \frac{1}{\pi \log 2} \frac{\mathrm{E}(\phi_p)}{\phi_p^2(z)}.$$

We shall not prove this conjecture. Since we have estimates for $\gamma_{2,1}(p)$, we still have estimates on the difficulty of doing analytic continuation even though they are not as good as the conjecture above. We shall soon see, however, that in many cases we still have *good* estimates, indeed, estimates which are not so far from what the conjecture would give us.

6.2.2 Estimates

Proposition 6.2.2 *Let $A \subseteq \Delta$ be a compact set and let $z_0 \notin A$. Let d be the distance between z_0 and A and let δ be the diameter of A, both measured in the hyperbolic metric. Then, if ϕ is a positive superharmonic function defined on Δ with value 0 on $\partial\Delta$ which is harmonic outside A we have*

$$\frac{\gamma_{1,2}(\phi)}{\phi^2(z_0)} \geq \frac{\log^2(\tanh(\delta/2))}{\log^2(\tanh(d/2))} \cdot \gamma_{1,2}(A).$$

Proof of Proposition 6.2.2:

From what we know about ϕ it follows that it can be written in the form

$$\phi(z) = \int_A \xi_w(z) d\mu(w)$$

where μ is a finite measure on A and $\xi_w(z) = |\log(|(z-w)/(\overline{w}z-1)|)|$. Let $\mu(A) = K$.

It is easy to check that for any $w \in A$, $\xi_w(z_0) \leq |\log(\tanh(d/2))|$. It is equally easy to check that if w and z are both in A, $\xi_w(z) \geq |\log(\tanh(\delta/2))|$. From the first inequality it follows that $\phi(z_0) \leq K|\log(\tanh(d/2))|$. From the second inequality it follows that, for any $z \in A$, $\phi(z) \geq K|\log(\tanh(\delta/2))|$.

This last inequality implies that $B = \phi^{-1}([K|\log(\tanh(\delta/2))|, +\infty))$ contains A as a subset. From proposition 4.2.4 it follows that $\gamma_{1,2}(\phi) \geq K^2\log^2(\tanh(\delta/2))\gamma_{1,2}(B) \geq K^2\log^2(\tanh(\delta/2))\gamma_{1,2}(A)$. Our result follows from this last inequality and our upper bound for $\phi(z_0)$.

$\boxed{6.2.2}$

We can easily see that

$$|\log(\tanh(x/2))| > 2e^{-x}$$

and that

$$|\log(\tanh(x/2))| \approx 2e^{-x}$$

is an excellent approximation for large x. This means the proposition could be stated as the following approximate inequality:

$$\frac{\gamma_{1,2}(\phi)}{\phi^2(z_0)} \gtrsim e^{2d-2\delta} \cdot \gamma_{1,2}(A).$$

In the usual case where $d > \delta$, we could confidently write '$\geq$' instead of '$\gtrsim$' since the errors are in our favor. This gives us the following corollary:

Corollary 6.2.3 *Let $A \subseteq \Delta$ be a compact set and let $z_0 \notin A$. Let d be the distance between z_0 and A and let δ be the diameter of A, both measured in the hyperbolic metric. Suppose $d > \delta$. Then, if ϕ is a positive superharmonic function defined on Δ with value 0 on $\partial\Delta$ which is harmonic outside A we have*

$$\frac{\gamma_{1,2}(\phi)}{\phi^2(z_0)} \geq e^{2d-2\delta} \cdot \gamma_{1,2}(A).$$

Proof of Corollary 6.2.3:

Above.

$\boxed{6.2.3}$

6.2.3 Examples

In this subsection we consider some explicit problems of doing analytic continuation and try to find hard how hard they are.

Example 6.2.4 *A is a round ball of radius R around the origin; z is a point a distance r from the origin, $r > R$.*

Let us define $R = \tanh(\rho/2)$ and $D = \tanh(r/2)$. ρ and D are the radius of A and the distance from the origin to z in the hyperbolic metric. Using the notation of proposition 6.2.2 we have $\delta = 2\rho$ and $d = D - \rho$; we shall define $d^- = \tanh(d/2)$.

We know that $\gamma_{1,2}(A) = 1/|\log R|$; we can easily prove that $\phi_A(z) = \log r/\log R$. From what we saw in the previous sections we can conclude that if we want to know $f(z)$ with precision ϵ we have to know f on A with precision δ where

$$\lim \frac{\log \delta}{\log \epsilon} = \frac{\log R}{\log d}.$$

We also know that if we want to find out the value of $f(z)$ with a precision ϵ we have to ask N_ϵ questions where

$$\lim \frac{N_\epsilon}{\log^2(\epsilon)} = \frac{1}{\log 2} \frac{|\log R|}{\log^2 d}$$

if we demand that f be known with equal precision on all of A. If, however, we are allowed to pay more attention to some parts of A and less to others, we have the inequality

$$\lim \frac{N_\epsilon}{\log^2(\epsilon)} \geq \frac{1}{\log 2} \frac{\log^2(\tanh(\delta/2))}{\log^2(\tanh(d/2))} \cdot \gamma_{1,2}(A).$$

If we assume $d > \delta$, which means, for a fixed R, if we assume r large,

$$\lim \frac{N_\epsilon}{\log^2(\epsilon)} \geq \frac{1}{\log 2} e^{2d-2\delta} \cdot \gamma_{1,2}(A).$$

The inequalities above may not seem very impressive but in the next example will make their analogues have a clear intuitive content.

Example 6.2.5 *Let the domain of functions now be the strip Σ of radius $\pi/2$ around the real axis. A is a round ball of radius $R < \pi/2$ around the origin; z is a point on the real line a distance D from the origin, $D > R$.*

In this case it is not so easy to compute $\gamma_{1,2}(A)$; it is enough to know that

$$\frac{2}{|\log(R/\pi)|} < \gamma_{1,2}(A) < \frac{1}{|\log(2R/\pi)|}.$$

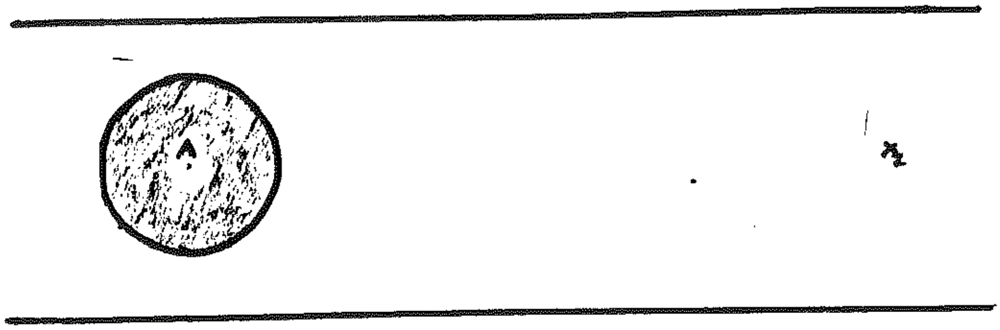

Figure 6.1: A is a ball inside Σ.

We can also see that for any $A' \subseteq A$

$$|\log(\tanh((D+R)/2))| < \frac{\phi_{A'}(z)}{\gamma_{1,2}(A')} < |\log(\tanh((D-R)/2))|;$$

we could follow this approach, but we prefer to use the results of the previous subsection. In order to get full generality we again use proposition 6.2.2. We have $d = D - R$ and $\delta = 2\log(\tan R + \sec R)$. This gives us

$$\frac{\gamma_{1,2}(\phi)}{\phi^2(z_0)} \geq \frac{\log^2(\tanh(\delta/2))}{\log^2(\tanh(d/2))} \cdot \gamma_{1,2}(A).$$

If we assume D large, we have

$$\frac{\gamma_{1,2}(\phi)}{\phi^2(z_0)} \geq e^{2d-2\delta} \cdot \gamma_{1,2}(A).$$

Using the method of asking questions discussed we have (assuming D large)

$$\lim \frac{N_\epsilon}{\log^2(\epsilon)} \geq \frac{1}{\log 2} e^{2d-2\delta} \cdot \gamma_{1,2}(A).$$

For a fixed R this inequality tells us that

$$N_\epsilon \geq K e^{2D} \log^2(\epsilon)$$

for sufficiently small ϵ where

$$K = \frac{e^{-2R-2\delta}\gamma_{1,2}(A)}{\log 2} \geq \frac{2e^{-2R-2\delta}}{\log 2|\log(R/\pi)|}.$$

In other words, N_ϵ grows quadratically with $\log(\epsilon)$ and *exponentially* with D.

We see that even for a moderate value D like 5 we would have

$$N_\epsilon > 22000K\log^2(\epsilon)$$

which means that in order to get $f(z)$ with a precision of 0.1 we need to ask more than $116000K$ questions; notice, however, that K may be small: for $R = 1$, this means more than 295 questions. If we ask instead for a precision of 0.01 we need to ask more than 1180 questions. If we keep $R = 1$, make $D = 10$ and ask for a precision of 0.1 we need to ask more than $6.5 \cdot 10^6$ questions, if we ask for a precision of 0.01 we need more than $2.6 \cdot 10^7$, if we take $D = 50$ and ask for precisions of 0.1 and 0.01 we need more than $3.6 \cdot 10^{41}$ and $1.44 \cdot 10^{42}$ questions respectively which is impossible to do in practice.

6.3 The Analytic Continuation Flow

6.3.1 The Entropy and Generalized Entropy of the Analytic Continuation Flow

In this subsection we define a flow corresponding to analytic continuation and show that this flow has infinite entropy. This already shows that analytic continuation is 'hard', but it only tells a small part of the story. We are going to see that the entropy is not merely infinite, but infinite by a comfortable margin. In other words, it is much harder to do analytic continuation than to follow an arbitrary flow of infinite entropy.

Let us begin by noticing that there is an analytic bijection between the disc Δ and the infinite strip around the real axis of 'radius' equal to $\pi/2$ which shall be denoted by Σ. (The reason for choosing $\pi/2$ is so that the Kobayashi metric at the origin coincides with the usual metric; this is not essential for us however.) This allows us to identify $\mathcal{M}$ to the space of analytic functions defined on Σ bounded in absolute value by 1 and $\mathcal{H}^\infty$ to the space of bounded analytic functions defined on Σ. We shall not introduce new notation for these spaces; we shall merely talk about $\mathcal{M}$ or $\mathcal{H}^\infty$ or even $\mathcal{H}$ and think of these spaces as having several 'interpretations', several 'models'.

With the interpretation of all these spaces as spaces of functions over Σ, we define the following action of $\mathbb{R}$ on $\mathcal{H}$:

$$\begin{array}{rccc} \theta : & \mathbb{R} \times \mathcal{H} & \rightarrow & \mathcal{H}. \\ & (t; f(z)) & \hookrightarrow & f(z+t) \end{array}$$

It is immediate to check that θ is indeed an action; it is in fact also an action on the subspaces $\mathcal{H}^\infty$ and $\mathcal{M}$. This action can of course be seen as a flow; we proceed now to compute the topological entropy of this flow. Before doing so, however, we have to give $\mathcal{M}$ a metric space stucture. We give it the metric d_A where $A \subseteq \Sigma$ is a compact set. The simplest situation, which the reader should have in mind before considering other cases, is that of A equal to a small disc around the origin.

Following the definition of topological entropy, we say that $B \subseteq \mathcal{M}$ is (t_0, ϵ)-generator if

$$\forall x \in \mathcal{M} \quad \exists y \in B \quad \forall t, 0 \leq t \leq t_0, \qquad d_A(\theta(t,x), \theta(t,y)) < \epsilon.$$

This is the same as saying that B is an ϵ-covering set of $\mathcal{M}$ for the metric $d_{A_{t_0}}$ where

$$A_{t_0} = \{z \in \Sigma | \exists t, 0 \leq t \leq t_0, z + t \in A\}.$$

Again following the definition of topological entropy we define $r(t_0, \epsilon)$ to be the size of the smallest (t_0, ϵ)-generator. We can easily see that $r(t_0, \epsilon) = NC_\epsilon(\mathcal{M}, d_{A_{t_0}})$. The topological entropy of θ is defined to be

$$h_{top}(\theta) = \lim_{\epsilon \to 0} \limsup_{t_0 \to \infty} \frac{1}{t_0} \log r(t_0, \epsilon) = \lim_{\epsilon \to 0} \limsup_{t_0 \to \infty} \frac{1}{t_0} \log NC_\epsilon(\mathcal{M}, d_{A_{t_0}}).$$

The conductivity theorem tells us that

$$\log NC_\epsilon(\mathcal{M}, d_{A_{t_0}}) \approx \mathbf{C} \cdot \log^2 \epsilon.$$

Using this approximation we would get

$$h_{top}(\theta) = \lim_{\epsilon \to 0} \limsup_{t_0 \to \infty} \frac{\mathbf{C}}{t_0} \log^2 \epsilon.$$

But it can easily be seen that

$$\lim_{t_0 \to \infty} \frac{\mathbf{C}(\Sigma; \partial\Sigma, \partial A_{t_0})}{t_0} = K > 0$$

from which it would follow that

$$h_{top}(\theta) = \lim_{\epsilon \to 0} K \log^2 \epsilon = +\infty.$$

The computations above were not done very carefully, however, and we should make them into a real proof. What we said above amounts to

$$\lim_{t_0\to\infty}\lim_{\epsilon\to 0}\frac{1}{\log^2\epsilon}\frac{1}{t_0}\log r(t_0,\epsilon)=K>0$$

and if from this we could conclude

$$\lim_{\epsilon\to 0}\lim_{t_0\to\infty}\frac{1}{\log^2\epsilon}\frac{1}{t_0}\log r(t_0,\epsilon)=K>0$$

that would give us what we want.

Notice that this would give us much more than the fact that the entropy is infinite; just as we defined the concept of generalized dimension we could define the concept of generalized entropy to be

$$\eta_r(\theta)=\lim_{\epsilon\to 0}\lim_{t_0\to\infty}\frac{1}{\log^r\epsilon}\frac{1}{t_0}\log r(t_0,\epsilon).$$

From this point of view we would have estimated $\eta_2(\theta)$, which would be positive, thus implying that $\eta_0(\theta)$, the usual entropy, is infinite. We formulate this unacheived conclusion as a conjecture (to be addressed in the future).

Conjecture 6.3.1 *If θ is the analytic continuation flow as defined above corresponding to the metric d_A we have*

$$\eta_2(\theta)=\lim_{\epsilon\to 0}\lim_{t_0\to\infty}\frac{1}{\log^2\epsilon}\frac{1}{t_0}\log r(t_0,\epsilon)=K(A)$$

where $K(A)$ denotes a numerical function of A assuming positive values.

In order not to leave the question entirely open we proceed in the next subsection to give a direct, elementary proof of the much weaker claim that the (usual) entropy is infinite.

6.3.2 The Entropy of the Analytic Continuation Flow is Infinite

Proposition 6.3.2 *If θ is the analytic continuation flow corresponding to d_A then*

$$h_{top}(\theta) = +\infty.$$

Proof of Proposition 6.3.2:

It is enough to consider the case $A = \{0\}$. We have to find an estimate for $\frac{1}{t}\log r(t,\epsilon)$. We know that $r(t,\epsilon) = NC_\epsilon(\mathcal{M}, d_{A_t})$. We have therefore to give an estimate for $NC_\epsilon(\mathcal{M}, d_{A_t})$. We shall do this by first estimating $NP_\epsilon(\mathcal{M}, d_{A_t})$ and then using the inequality

$$NP_\epsilon(S,d) \leq NC_{(1/3)\epsilon}(S,d)$$

from proposition 1.1.2. We have therefore to build a large packing set.

Let $h(z) = 2z^{-2}(1-\cos(z))$. We have $h(0) = 1$ and $f(2n\pi) = 0$ for any integer n. We also have

$$|h(z)| \leq 2(1+\cosh\frac{\pi}{2})$$

and

$$|h(a+bi)| \leq 2(1+\cosh\frac{\pi}{2})a^{-2}.$$

Let $N = 1 + \lfloor t/2\pi \rfloor$. We shall now construct our packing set with elements of the form

$$f(z) = \sum_{0\leq j<N} a_j h(z+2j\pi).$$

We can select the a_j arbitrarily with the condition $|a_j| < K$ and $f(z)$ will be in $\mathcal{M}$, where K is a constant whose value we do not need to know. If we select each one of them independently from an ϵ-packing set for the disc of radius K we get a packing set for $\mathcal{M}$. This gives us

$$NP_\epsilon(\mathcal{M}, d_{A_t}) \geq (K^2\epsilon^-2)^{(1+\lfloor t/2\pi \rfloor)}$$

which gives us (after some simple computations)

$$h_{top}(\theta) = +\infty.$$

$\boxed{6.3.2}$

The careful reader will have noticed that we in fact proved

$$\eta_1(\theta) > 0$$

which is stronger than the previous proposition but far weaker than the conjecture. This is, of course, to be expected given the great simplicity of the packing set constructed.

6.4 Explicit Methods for Doing Analytic Continuation

6.4.1 A Method

In this section we just take a quick look at some methods of doing analytic continuation. We do not attempt to be complete in any sense.

Example 6.4.1 *In this method we start with the power series expansion around a point and work by steps towards our objective.*

Start with the power series expansion of f around a 'known' point z_0. The power series gives an approximation for the function inside the disc of convergence. If we pick another point z_1 inside that disc, we can use this approximation in order to get an approximate power series expansion around z_1. We can repeat this process for z_2, z_3, $\ldots z_n$. Is this a good method for doing analytic continuation from z_0 to z_n?

In order to answer this question we first have to know how we use the approximate series expansion around z_i in order to approximate a series expansion around z_{i+1}. The first approximate series expansion is truncated at a certain point. In other words, it is a polynomial of degree N.

If we just rewrite this polynomial in term of $(z - z_{i+1})$ after n steps we still have the polynomial we had at the beginning. This will therefore never take us beyond the original disc of convergence. This is obviously not a very good method.

There is one known way out of this difficulty (see [4]). At each step, truncate the series from N to rN terms where r is a sufficiently small real parameter. In order to see more about this method, we refer the reader to the previouly mentioned book. We just want to observe that what we saw about analytic continuation indeed applies to this method: the loss of information is exponential with the distance travelled. The exact rate depends ont he value of r: from our results we can easily derive upper bounds for r. The reason why the necessary amount of information grows quadratically with $|\log \epsilon|$ may not be so obvious: this reflects the fact that the higher coefficients of the power series are harder and harder to obtain from the actual values of the function.

6.4.2 Another Method

Example 6.4.2 *In this method we start by using the Riemann mapping theorem to transform the domain of definition into a disc and then use series expansions.*

What we do here is reduce the entire problem to a situation very much like the one we have been studying. Use the Riemann mapping theorem to transform the domain of definition of f into the unit disc. If we do not know a priori of a domain where f is defined, this method becomes more complicated since we have to consider a sequence of neighbourhoods of the path along which analytic continuation is to be done. In any case, after we have transformed the domain into a disc we just use whatever series expansion applies to the situation. If we know the values of the function in a compact region A, we expand f in terms of the corresponding φ_n. If we know the germ of the function in one point, we take this point to the origin and use the corresponding Taylor series expansion. In any case, we see that a big distance between the known point or region and the point where we want the value of f corresponds to a great precision in determining the coefficients being required. The fact that the amount of information needed grows quadratically corresponds to the fact that if we want more precision, we need more coefficients *and* a greater precision in determining each coefficient.

Bibliography

[1] L. V. Ahlfors, *Conformal Invariants*, McGraw-Hill Book Company, 1973.

[2] V. D. Erohin, *Conform mappings of rings and the fundamental basis of the space of functions that are analytic in the elementary neighborhood of an arbitrary continuum*, Dokl. Akad. Nauk SSSR **120** (1958), 689-692.

[3] V. D. Erohin, *Tha asymptotic behavior of ϵ-entropy of analytic functions*, Dokl. Akad. Nauk SSSR **120** (1958), 949-952.

[4] P. Henrici, *Applied and Computational Complex Analysis*, a Wiley-Interscience publication.

[5] G. G. Lorentz, *Metric entropy and approximation*, Bull. Amer. Math. Soc. **72** (1966), 903-937.

[6] A. G. Vituškin, *Theory of transmission and processing of information*, Pergamon Press, New York, 1961.