arXiv:1312.1675v2 [math.GT] 17 Jan 2014

SPACES OF CURVES WITH CONSTRAINED CURVATURE ON FLAT SURFACES, I

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Abstract. Let S be a complete flat surface, such as the Euclidean plane. We obtain direct characterizations of the connected components of the space of all curves on S which start and end at given points in given directions, and whose curvatures are constrained to lie in a given interval, in terms of all parameters involved. Many topological properties of these spaces are investigated. Some conjectures of L. E. Dubins are proved.

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0. Introduction

To abbreviate the notation, we shall identify $\mathbf{R}^2$ with $\mathbf{C}$ throughout. A curve $\gamma\colon [0,1] \to \mathbf{C}$ is called *regular* if its derivative is continuous and never vanishes. Its *unit tangent*

$$\mathbf{t}_\gamma\colon [0,1] \to \mathbf{S}^1 \text{ is then given by } \mathbf{t}_\gamma(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|}.$$

Lifting γ to the unit tangent bundle $UT\mathbf{C} \equiv \mathbf{C} \times \mathbf{S}^1$, we obtain its *frame*

$$\Phi_\gamma\colon [0,1] \to \mathbf{C} \times \mathbf{S}^1, \quad \Phi_\gamma(t) = (\gamma(t), \mathbf{t}_\gamma(t)).$$

Let $P = (p,w),\ Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$ and consider the spaces of curves

$$\begin{aligned} \mathcal{S}(P,Q) &= \left\{\gamma\colon [0,1] \xrightarrow{C^r} \mathbf{C} :\ \gamma \text{ is regular, } \Phi_\gamma(0) = P \text{ and } \Phi_\gamma(1) = Q\right\} \\ \Omega UT\mathbf{C}(P,Q) &= \left\{\omega\colon [0,1] \xrightarrow{C^{r-1}} UT\mathbf{C} :\ \omega(0) = P \text{ and } \omega(1) = Q\right\} \end{aligned} \tag{1}$$

endowed with the C^r (resp. C^{r-1}) topology ($1 \leq r \in \mathbf{N}$). In 1956, S. Smale proved that the map

$$\Phi\colon \mathcal{S}(P,Q) \to \Omega UT\mathbf{C}(P,Q), \quad \gamma \mapsto \Phi_\gamma$$

is a weak homotopy equivalence (that is, it induces isomorphisms on homotopy groups). Actually, Smale's theorem ([11], thm. C) is much more general in that it holds for any manifold, not just $\mathbf{C}$.

2010 *Mathematics Subject Classification.* Primary: 53A04. Secondary: 53C42, 57N20.

Key words and phrases. Curve; Curvature; Dubins path; Flat surface; Plane; Topology of infinite-dimensional manifolds.

Using standard results on Banach manifolds which were discovered later, one can conclude that the spaces in (1) are in fact homeomorphic, and that the value of r is unimportant.

Given a regular curve γ, an *argument* of $\mathbf{t}_\gamma$ is a continuous function $\theta_\gamma\colon [0,1] \to \mathbf{R}$ such that $\mathbf{t}_\gamma = e^{i\theta_\gamma}$. The *total turning* of γ is defined to be $\theta_\gamma(1) - \theta_\gamma(0)$; note that this is independent of the choice of $\theta_\gamma(0)$. It is easy to see that $\Omega UT\mathbf{C}(P,Q)$ is homotopy equivalent to $\Omega\mathbf{S}^1(w,z)$. The latter possesses infinitely many connected components, one for each θ_1 satisfying $e^{i\theta_1} = z\bar{w} = zw^{-1}$, all of which are contractible. Therefore, the components of $\mathcal{S}(P,Q)$ are all contractible as well, and two curves in $\mathcal{S}(P,Q)$ lie in the same component if and only if they have the same total turning. This generalizes the Whitney-Graustein theorem of 1937 ([12], thm. 1), which states that two closed regular plane curves are homotopic through curves of the same type if and only if they have the same turning number.

The main purpose of this work is to investigate the topology of subspaces of $\mathcal{S}(P,Q)$ obtained by imposing constraints on the curvature of the curves.

Definition. Let $-\infty \le \kappa_1 < \kappa_2 \le +\infty$ and $2 \le r \in \mathbf{N}$. For $P = (p,w),\ Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$, define $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$ to be the set of all C^r regular curves $\gamma\colon [0,1] \to \mathbf{C}$ such that:

(i) $\Phi_\gamma(0) = P$ and $\Phi_\gamma(1) = Q$;
(ii) The curvature κ_γ of γ satisfies $\kappa_1 < \kappa_\gamma(t) < \kappa_2$ for each $t \in [0,1]$.

Let this set be furnished with the C^r topology.

Condition (i) means that γ starts at p in the direction of w and ends at q in the direction of z. In this notation, $\mathcal{S}(P,Q)$ becomes $\mathcal{C}^{+\infty}_{-\infty}(P,Q)$. The connected components of $\mathcal{C}^{+\kappa_0}_{-\kappa_0}(P,Q)$ ($\kappa_0 > 0$) were first studied by L. E. Dubins in [3]. His main result (thm. 5.3, slightly rephrased) implies that there may exist curves with the same total turning which are not homotopic within this space.

Theorem (Dubins)**.** *Let $x > 0$ and $Q_x = (x,1), O = (0,1) \in \mathbf{C} \times \mathbf{S}^1$. Let $\eta \in \mathcal{C}^{+1}_{-1}(O,Q_x)$ be the line segment $\eta\colon t \mapsto xt$. Then η and the concatenation of η with a figure eight curve lie in the same connected component of $\mathcal{C}^{+1}_{-1}(O,Q_x)$ if and only if $x > 4$. (See fig. 1.)*

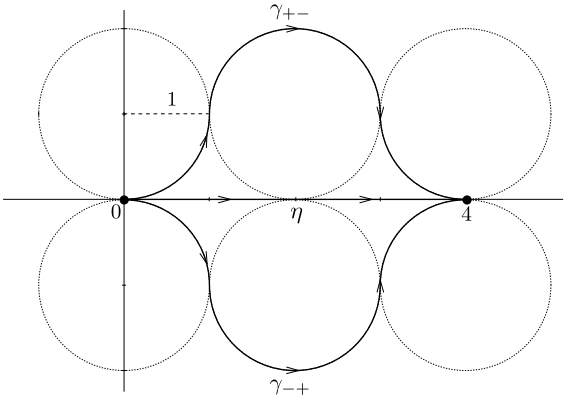


FIGURE 1. This shows a transitional configuration. Starting from η, for $x < 4$, it is not possible to create vertical tangents. For $x > 4$, after creating vertical tangents we can graft two long line segments (as in fig. 9, p. 29); then there is plenty of space.

Naturally, we always have the following decomposition of $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$ into closed-open subspaces:

$$\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q) = \bigsqcup_{\theta_1} \mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1),$$

where the union is over all $\theta_1 \in \mathbf{R}$ satisfying $e^{i\theta_1} = z\bar{w}$ and $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1) \subset \mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$ consists of those curves which have total turning equal to θ_1.

If $\kappa_1\kappa_2 \ge 0$, it will be shown that each $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$ is either empty or a contractible connected component of $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$.[†] If $\kappa_1\kappa_2 < 0$, then $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$ is never empty, and it is a contractible

[†]In determining the sign of $\kappa_1\kappa_2$, we adopt the convention that $0(\pm\infty) = 0$.

connected component provided that $|\theta_1| \geq \pi$. However, the remaining subspace $\mathcal{C}^{\kappa_2}_{\kappa_1}(P, Q; \theta_1)$ with $|\theta_1| < \pi$ may not be contractible, nor connected. It turns out that one can obtain simple and explicit characterizations of its components in terms of κ_1, κ_2, P and Q by using a homeomorphism with a space of the form $\mathcal{C}^{+1}_{-1}(P_0, Q_0; \theta_1)$ and an elementary geometric construction (see fig. 2).

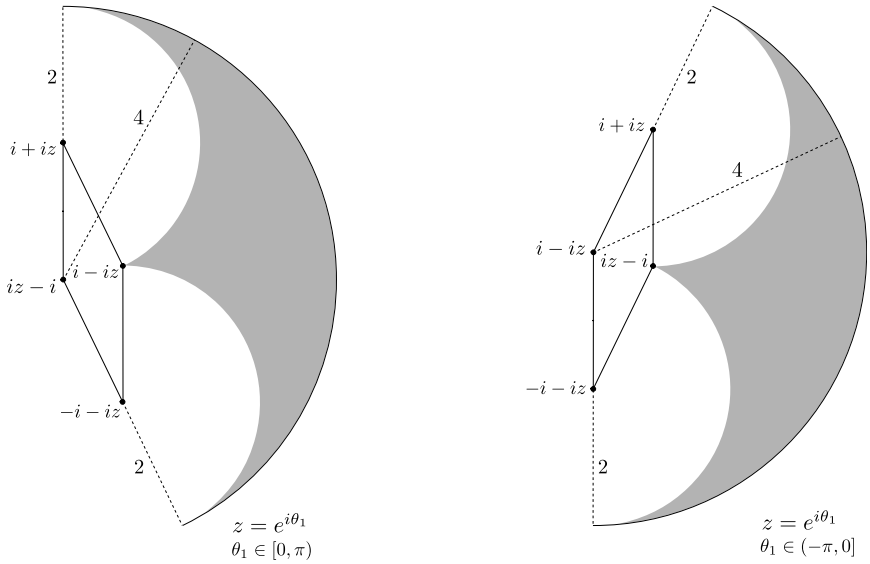


FIGURE 2. Let $\theta_1 \in \mathbf{R}$ be fixed, $z = e^{i\theta_1}$ and $Q = (q, z)$. Then $\mathcal{C}^{+1}_{-1}(Q; \theta_1)$ is disconnected if and only if $|\theta_1| < \pi$ and q lies in the gray region. The region contains the arc of circle of radius 4, but not the arcs of circle of radius 2. Figure (a) depicts the case $\theta_1 \in [0, \pi)$, and (b) the case $\theta_1 \in (-\pi, 0]$ (here $\theta_1 \approx \pm 26°$). The theorem of Dubins stated above corresponds to $\theta_1 = 0$.

This paper is close in spirit to Dubins' [3], and here we shall settle some of his conjectures. (It is *not* assumed, however, that the reader is familiar with his work.) In the sequel [9] to this article, we investigate the homeomorphism class of $\mathcal{C}^{\kappa_2}_{\kappa_1}(P, Q)$. Surprisingly, for $\kappa_1\kappa_2 < 0$, the "exotic" subspace containing curves of least total turning may be homotopy equivalent to an n-sphere.

Outline of the sections. Many useful constructions, such as the concatenation of elements of $\mathcal{C}^{\kappa_2}_{\kappa_1}(P, Q)$ and $\mathcal{C}^{\kappa_2}_{\kappa_1}(Q, R)$, yield curves which need not be of class C^2. To avoid having to smoothen curves all the time, we work with curves which have a continuously varying unit tangent at all points, but whose curvatures are defined only almost everywhere. The resulting spaces are denoted by $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$. In §1 we prove that the set inclusion $\mathcal{C}^{\kappa_2}_{\kappa_1}(P, Q) \to \mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ is a homotopy equivalence with dense image and that these spaces are homeomorphic. In this section, all concepts and results are formulated over an arbitrary complete oriented surface S. For $u, v \in UTS$, it is shown that the corresponding spaces $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ and $\mathcal{C}^{\kappa_2}_{\kappa_1}(u, v)$ of curves on S are always nonempty if S is compact (and connected); this is also true if S is flat, but false if S is the hyperbolic plane.

Starting with §2, we work almost exclusively with plane curves. Let $O = (0, 1) \in \mathbf{C} \times \mathbf{S}^1$ denote the canonical element of $UT\mathbf{C}$. A proper Euclidean motion is uniquely determined by where it maps O. Thus, $UT\mathbf{C}$ carries a natural Lie group structure. This allows us to give an alternative definition of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ and to view the frame of a curve as a one-parameter family of isometries.

Let us denote $\mathcal{C}^{\kappa_2}_{\kappa_1}(O, Q)$ simply by $\mathcal{C}^{\kappa_2}_{\kappa_1}(Q)$. Using Euclidean motions, dilatations and a construction called *normal translation* (see fig. 3 on p. 14), we obtain in §3 an explicit homeomorphism between any space $\mathcal{C}^{\kappa_2}_{\kappa_1}(P_0, Q_0)$ and a space of one of the following types: $\mathcal{C}^{+\infty}_0(Q)$, $\mathcal{C}^{+\infty}_1(Q)$ or $\mathcal{C}^{+1}_{-1}(Q)$, according as $\kappa_1\kappa_2 = 0$, $\kappa_1\kappa_2 > 0$ or $\kappa_1\kappa_2 < 0$, respectively. Moreover, this homeomorphism preserves the total turning of curves up to sign. Among these three, $\mathcal{C}^{+1}_{-1}(Q)$ has the most interesting topological properties.

We call a regular curve $\gamma\colon [0,1] \to \mathbf{C}$ *condensed*, *critical* or *diffuse*, according as its *amplitude*

$$\omega = \max_{t\in[0,1]} \theta_\gamma(t) - \min_{t\in[0,1]} \theta_\gamma(t)$$

satisfies $\omega < \pi$, $\omega = \pi$ or $\omega > \pi$, respectively. Let $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$ and θ_1 be such that $e^{i\theta_1} = z$. Let $\mathcal{U}_c$, $\mathcal{U}_d \subset \mathcal{C}^{+1}_{-1}(Q;\theta_1)$ denote the subspaces consisting of all condensed (resp. diffuse) curves. Both are open and $\mathcal{U}_d \neq \emptyset$, since we may always concatenate a curve in $\mathcal{C}^{+1}_{-1}(Q;\theta_1)$ with a curve of total turning 0 (an eight curve, as in fig. 8(e) on p. 26). Clearly, $\mathcal{U}_c$ must be empty if $|\theta_1| \geq \pi$, but it may also be empty otherwise, depending on Q. We determine exactly when this occurs in §4.

A condensed curve may be viewed as the graph of a function with respect to some axis. This leads to a direct, albeit involved, proof that $\mathcal{U}_c$ is contractible, when nonempty. In fact, if the curvatures are allowed to be discontinuous and to take values in the closed interval $[-1, 1]$, then one can exhibit a contraction of the subspace of condensed curves to the unique curve of minimal length (*Dubins path*) in the corresponding space. This is also done in §4.

In §5 an indirect proof that $\mathcal{U}_d$ is contractible is obtained. If γ is diffuse, then we can "graft" straight lines segment onto γ, as illustrated in fig. 9, p. 29. Such a segment can be deformed so that in the end an eight curve of large radius traversed a number n of times has been attached to it. These eights are then spread along the curve, as in fig. 8(f). If $n \in \mathbf{N}$ is large enough, then the whole process can be carried out within $\mathcal{C}^{+1}_{-1}(Q)$. The result is a curve which is so loose that the constraints on the curvature become irrelevant.

Using elementary geometry, we decide in §6 when the set $\mathcal{T}$ of all critical curves in $\mathcal{C}^{+1}_{-1}(Q;\theta_1)$ is empty. The main result in this section is that $\mathcal{T} = \partial\mathcal{U}_c = \partial\mathcal{U}_d$. When $\mathcal{T} \neq \emptyset$, a finer analysis of how $\partial\mathcal{U}_c$ and $\partial\mathcal{U}_d$ fit together is required to determine the homeomorphism class of $\mathcal{C}^{+1}_{-1}(Q;\theta_1)$. This problem will be treated in [9].

In §7 we obtain various characterizations of the connected components of $\mathcal{C}^{+1}_{-1}(Q;\theta_1)$. Perhaps the simplest one is the following: this space is disconnected if and only if $|\theta_1| < \pi$ and q lies in the region illustrated in fig. 2, or, equivalently, its subset $\mathcal{T}$ is empty, but $\mathcal{U}_c$ is not. In this case, it has exactly two components, $\mathcal{U}_c$ and $\mathcal{U}_d$, which are contractible. As mentioned previously, this is sufficient to determine explicitly the components of any space $\mathcal{C}^{\kappa_2}_{\kappa_1}(P_0, Q_0)$ with $\kappa_1\kappa_2 < 0$.

In §8 it is established that when $\kappa_1\kappa_2 \geq 0$, the space $\mathcal{C}^{\kappa_2}_{\kappa_1}(P, Q)$ has one connected component for each realizable total turning, and they are all contractible. However, not every total turning is possible, because if γ is a curve in such a space and θ_γ is an argument of $\mathbf{t}_\gamma$, then θ_γ must be monotone. One can determine the realizable total turnings in terms of all parameters involved using normal translation and elementary geometry. In fact, if curvatures in $[\kappa_1, \kappa_2]$ are permitted, then our method may be used to find a canonical curve which attains the minimal or maximal total turning.

In §9 these results are extended to spaces of curves with constrained curvature on any complete flat surface S (orientable or not), using the fact that if S is connected then it must be the quotient of $\mathbf{C}$ by a group of isometries.

Even though we have imposed that the curvatures should lie in an open interval, the main results obtained here have analogues for the spaces $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P, Q)$ (defined in §2), where the curvature is constrained to lie in $[\kappa_1, \kappa_2]$. For $\kappa_1 = -\kappa_2$, this is the class of spaces that Dubins actually worked with in [2] and [3]. The necessary modifications in the statements and proofs are explained in §10, where we also discuss a few conjectures about curves of minimal length.

1. Spaces of curves with constrained curvature on a complete surface

Basic terminology. Although we shall be primarily concerned with spaces of plane curves, it is desirable to treat the fundamental concepts in a more general setting.

Let S be a complete and oriented surface. Here a *surface* is simply a smooth Riemannian 2-manifold (not necessarily a submanifold of some Euclidean space). A *curve* on S is a continuous map $\gamma\colon [a, b] \to S$. We shall say that γ is *regular* if it has a continuous derivative which never vanishes. In this case, its *unit tangent* $\mathbf{t} = \mathbf{t}_\gamma$ is the lift of γ to the unit tangent bundle UTS of S:

$$\mathbf{t}\colon [a, b] \to UTS, \quad \mathbf{t}(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|}.$$

Let $\mathrm{pr}\colon TS \to S$ denote the tangent bundle of S. Define a bundle morphism $J\colon TS \to TS$ ("multiplication by i") by the following conditions:

(i) J preserves norms;
(ii) For $u \neq 0$, the pair $(u, J(u))$ is a positively oriented orthogonal basis for $TS_{\mathrm{pr}(u)}$.

The *unit normal* to a regular curve γ is the map $\mathbf{n} = \mathbf{n}_\gamma\colon [a,b] \to UTS$ given by $\mathbf{n} = J \circ \mathbf{t}$.

The *arc-length parameter* s of γ is defined by $s(t) = \int_a^t |\dot\gamma(\tau)|\, d\tau$ $(t \in [a,b])$ and $L = \int_a^b |\dot\gamma(\tau)|\, d\tau$ is the *length* of γ. Assuming γ is twice differentiable, its *curvature* $\kappa = \kappa_\gamma$ is given by

$$\kappa(s) = \Big\langle \frac{D\mathbf{t}}{ds}(s), \mathbf{n}(s) \Big\rangle \quad (s \in [0, L]).$$

Here D denotes covariant differentiation (along γ). In terms of a general parameter t,

$$\kappa = \frac{1}{|\dot\gamma|}\Big\langle \frac{D\mathbf{t}}{dt}, \mathbf{n} \Big\rangle = \frac{1}{|\dot\gamma|^2}\Big\langle \frac{D\dot\gamma}{dt}, \mathbf{n} \Big\rangle. \tag{2}$$

Notice that the curvature at each point is not altered by an orientation-preserving reparametrization of the curve, while its sign changes if the reparametrization is orientation-reversing.

Spaces of curves. We begin by introducing spaces of curves of class C^r.

(1.1) Definition. Let $-\infty \le \kappa_1 < \kappa_2 \le +\infty$ and S be a complete oriented surface. For $u, v \in UTS$, define $\mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)^r$ to be the set, endowed with the C^r topology (for some $r \in \{2, 3, \dots, \infty\}$), of all C^r regular curves $\gamma\colon [0,1] \to S$ such that:

(i) $\mathbf{t}_\gamma(0) = u$ and $\mathbf{t}_\gamma(1) = v$;
(ii) $\kappa_1 < \kappa_\gamma(t) < \kappa_2$ for each $t \in [0,1]$.

Let $\mathrm{pr}\colon UTS \to S$ denote the natural projection. Condition (i) states that γ starts at $\mathrm{pr}(u)$ in the direction of u and ends at $\mathrm{pr}(v)$ in the direction of v. In condition (ii), we could have required instead that κ_γ lie in some prescribed open subset of $\mathbf{R}$; the resulting space is the disjoint union of spaces of the type above. It will follow from (1.16) that the value of r is not important, in the sense that different choices of r yield spaces which are homeomorphic. When this does not lead to confusion, we shall denote $\mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)^r$ simply by $\mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)$.

(1.2) Lemma. *Let $(\kappa_1, \kappa_2) \subset (\bar\kappa_1, \bar\kappa_2)$. Then the set inclusion $\mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)^r \to \mathcal{C}^{\bar\kappa_2}_{\bar\kappa_1}(u,v)^r$ is a topological embedding for any $r \ge 2$, $u, v \in UTS$.* □

(1.3) *Remark.* It is natural to ask why we have required that the curvatures lie in open, instead of closed, intervals. The reason is that, for our purposes, the former condition is more convenient to work with. As an example, consider the space of all C^2 regular curves $\gamma\colon [0,1] \to \mathbf{C}$ whose curvatures are restricted to lie in $[-1,1]$ and which satisfy $\mathbf{t}_\gamma(0) = (1,i)$, $\mathbf{t}_\gamma(1) = (i,-1)$, where we have identified $UT\mathbf{C}$ with $\mathbf{C} \times \mathbf{S}^1$. The arc α of the unit circle given by $t \mapsto \exp(\frac{\pi i}{2} t)$ $(t \in [0,1])$ is a curve in this space. In fact, it is not hard to see that α is an isolated point, i.e., its connected component does not contain any other curve. In contrast, the spaces $\mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)^r$ are Banach manifolds (for $r \neq \infty$). Still, some useful constructions, such as the concatenation of curves, lead out of this class of spaces. To avoid having to smoothen curves all the time, we shall work with another class of spaces, which possess the additional advantage of being Hilbert manifolds.

Suppose that $\gamma\colon [0,1] \to S$ is a C^2 regular curve. Then γ and $\mathbf{t}_\gamma = \mathbf{t}\colon [0,1] \to TS$ satisfy:

$$\begin{cases} \dot\gamma = \sigma \mathbf{t} \\ \frac{D\mathbf{t}}{dt} = \sigma\kappa\,(J \circ \mathbf{t}) \end{cases} \qquad \text{and} \quad \mathbf{t}(0) = u \in TS. \tag{3}$$

Here $\sigma\colon [0,1] \to \mathbf{R}^+$ is the speed of γ, $\sigma(t) = |\dot\gamma(t)|$, and κ its curvature. Thus, γ is uniquely determined by $u \in UTS$ and the functions σ, κ. Our favorite class of spaces is obtained by relaxing the conditions that the functions σ and κ be of class C^1 and C^0, respectively.

Let $h\colon (0, +\infty) \to \mathbf{R}$ be the smooth diffeomorphism

$$h(t) = t - t^{-1}.$$

For each pair $\kappa_1 < \kappa_2 \in \mathbf{R}$, let $h_{\kappa_1,\,\kappa_2}\colon (\kappa_1, \kappa_2) \to \mathbf{R}$ be the smooth diffeomorphism

$$h_{\kappa_1,\,\kappa_2}(t) = (\kappa_1 - t)^{-1} + (\kappa_2 - t)^{-1}$$

and, similarly, set

$$\begin{aligned} h_{-\infty,+\infty}&\colon \mathbf{R} \to \mathbf{R} && h_{-\infty,+\infty}(t) = t \\ h_{-\infty,\kappa_2}&\colon (-\infty, \kappa_2) \to \mathbf{R} && h_{-\infty,\kappa_2}(t) = t + (\kappa_2 - t)^{-1} \\ h_{\kappa_1,+\infty}&\colon (\kappa_1, +\infty) \to \mathbf{R} && h_{\kappa_1,+\infty}(t) = t + (\kappa_1 - t)^{-1}. \end{aligned}$$

Notice that all of these functions are monotone increasing, hence so are their inverse functions. In all that follows, $\mathbf{E}$ denotes the separable Hilbert space $L^2[0,1] \times L^2[0,1]$.

(1.4) Definition. Let $-\infty \le \kappa_1 < \kappa_2 \le +\infty$ and S be a complete and oriented surface. A curve $\gamma\colon [0,1] \to S$ is (κ_1, κ_2)-*admissible* if there exists $(\hat\sigma, \hat\kappa) \in \mathbf{E}$ such that γ satisfies (3) with

$$\sigma = h^{-1} \circ \hat\sigma \quad \text{and} \quad \kappa = h^{-1}_{\kappa_1,\,\kappa_2} \circ \hat\kappa. \tag{4}$$

When it is not important to keep track of the bounds κ_1, κ_2, we will simply say that γ is *admissible*.

The system (3) has a unique solution for any $(\hat\sigma, \hat\kappa) \in \mathbf{E}$ and $u \in TS$. To see this, we use coordinate charts for TS derived from charts for S and apply thm. C.3 on p. 386 of [13] to the resulting differential equation, using the fact that S is smooth and $\sigma, \kappa \in L^2[0,1] \subset L^1[0,1]$. The hypothesis that S is complete is used to ensure that the solution is defined over all of $[0,1]$. The resulting maps $\gamma\colon [0,1] \to S$ and $\mathbf{t}\colon [0,1] \to TS$ are continuous (see p. 385 of [13]), and so is $\mathbf{n} := J \circ \mathbf{t}$. Using that $\langle \mathbf{t}, \mathbf{n} \rangle \equiv 0$ and differentiating, we obtain:

$$|\mathbf{t}(t)| = |\mathbf{n}(t)| = |u| \quad \text{for all } t \in [0,1] \quad \text{and} \quad \frac{D\mathbf{n}}{dt} = -\sigma\kappa\,\mathbf{t}.$$

Therefore, if $u \in UTS$, then $\sigma = |\dot\gamma|$, $\mathbf{t}_\gamma = \mathbf{t}$, $\mathbf{n}_\gamma = \mathbf{n}$ and κ is the curvature of γ, as expected.

(1.5) Definition. Let $-\infty \le \kappa_1 < \kappa_2 \le +\infty$ and S be a complete oriented surface. For $u \in UTS$, define $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot)$ to be the set of all (κ_1, κ_2)-admissible curves $\gamma\colon [0,1] \to S$ with $\mathbf{t}_\gamma(0) = u$. This set is identified with $\mathbf{E}$ via the correspondence $\gamma \leftrightarrow (\hat\sigma, \hat\kappa)$, thus furnishing $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot)$ with a trivial Hilbert manifold structure.[†]

Remark. Although $\dot\gamma = \sigma\mathbf{t}$ is, in general, defined only almost everywhere on $[0,1]$, if we reparametrize γ by arc-length then it becomes a regular curve, since $\gamma' = \mathbf{t}$ is continuous. Instead of thinking of γ as a pair of L^2 functions, it is more helpful to regard γ as a regular curve whose curvature is defined only a.e.. In fact, all of the concrete examples considered below are piecewise C^2 curves.

(1.6) Lemma. *Let $-\infty \le \kappa_1 < \kappa_2 \le +\infty$, S be a complete oriented surface and $u \in UTS$. Define $F\colon \mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot) \to UTS$ by $\gamma \mapsto \mathbf{t}_\gamma(1)$. Then F is a submersion. Consequently, it is an open map.*

The proof of this result will be given on p. 13.

(1.7) Definition. Let $-\infty \le \kappa_1 < \kappa_2 \le +\infty$, S be a complete oriented surface and $u, v \in UTS$. Define $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ to be the subspace of $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot)$ consisting of all $\gamma \in \mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot)$ such that $\mathbf{t}_\gamma(1) = v$.

It follows from (1.6) that $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ is a closed submanifold of codimension 3 in $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot) \equiv \mathbf{E}$, provided it is nonempty.

(1.8) *Remark.* Take $S = \mathbf{H}^2$ (the hyperbolic plane) and let u be any element of $UT\mathbf{H}^2$. Then it is not hard to see that $\mathcal{L}^{+1}_{-1}(u, u) = \emptyset$, that is, there are no closed curves in $\mathbf{H}^2$ whose curvatures lie in $(-1, 1)$ a.e.. Thus, even though the map F of (1.6) is a submersion, it need not be surjective. In contrast, $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v) \neq \emptyset$ for any choice of $\kappa_1 < \kappa_2$ and $u, v \in UTS$ if $S = \mathbf{R}^2$ or $S = \mathbf{S}^2$. This difference may ultimately be attributed to the fact that in $\mathbf{R}^2$ and $\mathbf{S}^2$, curves of constant curvature κ are always closed (indeed, they are exactly the circles), but this is only true for $\mathbf{H}^2$ when $|\kappa| > 1$.

The proof that $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v) \neq \emptyset$ when $S = \mathbf{R}^2$ will be postponed to §5. That $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v) \neq \emptyset$ when $S = \mathbf{S}^2$ follows from the next result.

[†]The '$\mathcal{L}$' is intended to remind one of L^2 functions.

(1.9) Proposition. *Let S be a compact connected oriented surface. Then, for any choice of $\kappa_1 < \kappa_2$ and $u, v \in UTS$, $\mathcal{L}^{\kappa_2}_{\kappa_1}(u,v) \neq \emptyset$ and $\mathcal{C}^{\kappa_2}_{\kappa_1}(u,v) \neq \emptyset$*

Proof. We shall prove the proposition for $\mathcal{L}^{\kappa_2}_{\kappa_1}(u,v)$. For $\mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)$, the assertion can then be derived from (1.14). Let $\kappa_1 < \kappa_2$ be fixed. For $u, v \in UTS$, write $u \prec v$ if $\mathcal{L}^{\kappa_2}_{\kappa_1}(u,v) \neq \emptyset$. Notice that $\prec$ is transitive. Let

$$M_u = \big\{v \in UTS : u \prec v\big\}, \quad N_u = \big\{v \in UTS : u \prec v \text{ and } v \prec u\big\}.$$

It is clear from the definition that $M_u \neq \emptyset$ for all u, and (1.6) implies that M_u and N_u are open subsets of UTS. Moreover, the family $(M_u)_{u \in UTS}$ covers UTS. Indeed, given v, we can find u such that $\mathcal{L}^{-\kappa_1}_{-\kappa_2}(-v,-u) \neq \emptyset$. Reversing the orientation of a curve in the latter set, we establish that $\mathcal{L}^{\kappa_2}_{\kappa_1}(u,v) \neq \emptyset$, that is, $v \in M_u$.

Since UTS is compact, it can be covered by finitely many of the M_u. Let $UTS = M_{u_1} \cup \cdots \cup M_{u_m}$ be a minimal cover. We claim that $m = 1$.

Assume that $m > 1$. If $u_i \prec u_j$ then $M_{u_i} \supset M_{u_j}$, and therefore by minimality $i = j$. Since $u_i \in UTS$ and $u_i \notin M_{u_j}$ for $j \neq i$, we deduce that $u_i \in M_{u_i}$ for each i. The open sets N_{u_i} are thus nonempty and disjoint. On the other hand, every M_{u_i} must intersect some M_{u_j} with $j \neq i$, as UTS is connected. Choose $i \neq j$ such that $M_{u_i} \cap M_{u_j} \neq \emptyset$. It is easy to see that $M_{u_i} \cap M_{u_j}$ must be disjoint from both N_{u_i} and N_{u_j}. Thus, if V is the interior of $M_{u_i} \smallsetminus N_{u_i}$, then $V \neq \emptyset$. We will obtain a contradiction from this. By definition, there exist $u_* \in N_{u_i}$, $v_* \in V$ and a curve $\gamma_* \in \mathcal{L}^{\kappa_2}_{\kappa_1}(u_*, v_*)$. Let $\gamma_* \colon [0, L] \to S$ be parametrized by arc-length and $\kappa_* \colon [0, L] \to (\kappa_1, \kappa_2)$ denote its curvature.

Recall that the cotangent bundle of S has a natural volume form. The tangent bundle TS then inherits a volume form from the Riemannian structure of S. It is well known that the geodesic flow preserves volume in TS. Since S is oriented, given $\theta \in \mathbf{R}$ we may define a volume-preserving bundle automorphism on TS by $w \mapsto \cos\theta w + \sin\theta J(w)$. Let Y_0 and Z be the vector fields on TS corresponding to the geodesic flow and to counterclockwise rotation, respectively. Then for any $\kappa \in \mathbf{R}$, the vector field $Y_\kappa = Y_0 + \kappa Z$ defines a volume-preserving flow on UTS; the projections of its orbits on S are curves parametrized by arc-length of constant curvature κ.

By definition, the open set V is forward-invariant under each of these flows, for $\kappa \in (\kappa_1, \kappa_2)$. Define a map $G \colon UTS \to UTS$ as follows: Given $u \in UTS$, $G(u) = \Phi_\eta(L)$, where $\eta \colon [0, L] \to S$ is the unique curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot)$, parametrized by arc-length, whose curvature is κ_*. Then G must be volume-preserving, $G(V) \subset V$, but there exists a neighborhood of u_* contained in N_{u_i} which is taken by G to a neighborhood of v_* contained in V, a contradiction.

We conclude that $m = 1$, so that $M_{u_1} = UTS$. Furthermore, $M_{u_1} \smallsetminus N_{u_1}$ must have empty interior by the above argument. Hence, N_{u_1} is a dense open set in UTS. Let $u, v \in UTS$ be given. Since M_u is open, there exists $v_1 \in N_{u_1}$ such that $u \prec v_1$. Since $v_1 \prec u_1$ and $u_1 \prec v$, we deduce that $u \prec v$, that is, $\mathcal{L}^{\kappa_2}_{\kappa_1}(u,v) \neq \emptyset$. □

If $(\kappa_1, \kappa_2) \subset (\bar\kappa_1, \bar\kappa_2)$ and $\gamma \in \mathcal{L}^{\kappa_2}_{\kappa_1}(u,v)$, then we can also consider γ as a curve in $\mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u,v)$. This yields an inclusion $j \colon \mathcal{L}^{\kappa_2}_{\kappa_1}(u,v) \to \mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u,v)$. However, the topology of the former space is strictly finer (i.e., has more open sets) than the topology induced by the latter.

(1.10) Lemma. *Let $(\kappa_1, \kappa_2) \subset (\bar\kappa_1, \bar\kappa_2)$, S be a complete oriented surface and $u \in UTS$. Then*

$$(\hat\sigma, \hat\kappa) \mapsto \big(\hat\sigma, h_{\bar\kappa_1, \bar\kappa_2} \circ h^{-1}_{\kappa_1, \kappa_2} \circ \hat\kappa\big) \tag{5}$$

defines a continuous injection $j \colon \mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot) \to \mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u, \cdot)$. The actual curves on S corresponding to $(\hat\sigma, \hat\kappa) \in \mathcal{L}^{\kappa_2}_{\kappa_1}(u, \cdot)$ and to $j(\hat\sigma, \hat\kappa) \in \mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u, \cdot)$ are the same. However, j is not a topological embedding unless $\bar\kappa_1 = \kappa_1$ and $\bar\kappa_2 = \kappa_2$.

Proof. The curve γ corresponding to $(\hat\sigma, \hat\kappa) \in \mathcal{L}^{\kappa_2}_{\kappa_1}(u,v)$ is obtained as the solution of (3) with

$$\sigma = h^{-1} \circ \hat\sigma \quad \text{and} \quad \kappa = h^{-1}_{\kappa_1,\, \kappa_2} \circ \hat\kappa.$$

The curve η corresponding to the right side of (5) in $\mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u, \cdot)$ is the solution of (3) with

$$\sigma = h^{-1} \circ \hat\sigma \quad \text{and} \quad \kappa = h^{-1}_{\bar\kappa_1,\, \bar\kappa_2} \circ \big(h_{\bar\kappa_1, \bar\kappa_2} \circ h^{-1}_{\kappa_1, \kappa_2} \circ \hat\kappa\big) = h^{-1}_{\kappa_1,\, \kappa_2} \circ \hat\kappa.$$

By uniqueness of solutions of (3), $\gamma = \eta$. In particular, j is injective.

Set $g = h_{\bar\kappa_1,\bar\kappa_2} \circ h_{\kappa_1,\kappa_2}^{-1}$. Observe that

$$\lim_{t\to+\infty} g'(t) = \begin{cases} 1 & \text{if } \bar\kappa_2 = \kappa_2; \\ 0 & \text{otherwise.} \end{cases} \qquad \text{Similarly,} \qquad \lim_{t\to-\infty} g'(t) = \begin{cases} 1 & \text{if } \bar\kappa_1 = \kappa_1; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $|g'|$ is bounded on $\mathbf{R}$. Consequently, there exists $C > 0$ such that

$$\|g \circ h_1 - g \circ h_2\|_2 \le C \, \|h_1 - h_2\|_2 \quad \text{for any } h_1,\, h_2 \in L^2[0,1].$$

We conclude that $j\colon (\hat\sigma, \hat\kappa) \mapsto (\hat\sigma, g \circ \hat\kappa)$ is continuous.

Suppose now that $(\kappa_1, \kappa_2) \subsetneq (\bar\kappa_1, \bar\kappa_2)$. No generality is lost in assuming that $\kappa_2 < \bar\kappa_2$. Let

$$m = g(0) \quad \text{and} \quad M = g(+\infty) = h_{\bar\kappa_1,\bar\kappa_2}(\kappa_2).$$

Define a sequence of L^2 functions $\hat\kappa_n\colon [0,1] \to \mathbf{R}$ by:

$$\hat\kappa_n(t) = \begin{cases} n & \text{if } t \in \left[\frac12 - \frac{1}{2n}\,,\, \frac12 + \frac{1}{2n}\right]; \\ 0 & \text{otherwise.} \end{cases} \qquad (n \in \mathbf{N},\ t \in [0,1]).$$

Since g is the composite of increasing functions, $g(t) < g(+\infty) = M$ for any $t \in \mathbf{R}$. Therefore,

$$|g \circ \hat\kappa_n(t) - m| \begin{cases} < M - m & \text{if } t \in \left[\frac12 - \frac{1}{2n}\,,\, \frac12 + \frac{1}{2n}\right]; \\ = 0 & \text{otherwise.} \end{cases}$$

Hence, $\|\hat\kappa_n\|_2 = \sqrt{n} \to +\infty$ as n increases, while $\|g \circ \hat\kappa_n - m\|_2 \le \frac{M-m}{\sqrt{n}} \to 0$. We conclude that $j\colon \mathcal{L}^{\kappa_2}_{\kappa_1}(u,\cdot) \to \mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u,\cdot)$ is not a topological embedding. The argument may be modified to prove that $j\colon \mathcal{L}^{\kappa_2}_{\kappa_1}(u,v) \to \mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u,v)$ is also not an embedding for any v, but we do not need this. □

(1.11) Lemma. *Let S be a complete oriented surface, $(\kappa_1,\kappa_2) \subset (\bar\kappa_1,\bar\kappa_2)$ and $u, v \in UTS$. Then*

$$j\colon \mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)^r \to \mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u,v), \quad \gamma \mapsto (\hat\sigma_\gamma, \hat\kappa_\gamma), \quad \hat\sigma_\gamma = h \circ |\dot\gamma| \quad \text{and} \quad \hat\kappa_\gamma = h^{\bar\kappa_2}_{\bar\kappa_1} \circ \kappa_\gamma \tag{6}$$

is a continuous injection, but not an embedding, for all $r \ge 2$. Moreover, the actual curve on S corresponding to $j(\gamma) \in \mathcal{L}^{\bar\kappa_2}_{\bar\kappa_1}(u,v)$ is γ itself.

Proof. The proof is very similar to that of (1.10). □

The following lemmas contain all the results on infinite-dimensional manifolds that we shall use.

(1.12) Lemma. *Let $\mathcal{M}$, $\mathcal{N}$ be (infinite-dimensional) Banach manifolds. Then:*[†]

(a) *$\mathcal{M}$ is locally path-connected. In particular, its connected and path components coincide.*
(b) *If $\mathcal{M}$, $\mathcal{N}$ are weakly homotopy equivalent, then they are in fact homeomorphic (diffeomorphic if $\mathcal{M}$, $\mathcal{N}$ are Hilbert manifolds).*
(c) *Let $\mathbf{E}$ and $\mathbf{F}$ be separable Banach spaces. Suppose $i\colon \mathbf{F} \to \mathbf{E}$ is a bounded, injective linear map with dense image and $\mathcal{M} \subset \mathbf{E}$ is a smooth closed submanifold of finite codimension. Then $\mathcal{N} = i^{-1}(\mathcal{M})$ is a smooth closed submanifold of $\mathbf{F}$ and $i\colon \mathcal{N} \to \mathcal{M}$ is a homotopy equivalence.*

Proof. Part (a) is obvious. Part (b) follows from thm. 15 in [7] and cor. 3 in [4]. Part (c) is thm. 2 in [1]. □

(1.13) Lemma. *Let $\mathbf{E}$ be a separable Hilbert space, $D \subset \mathbf{E}$ a dense vector subspace, $L \subset \mathbf{E}$ a submanifold of finite codimension and U an open subset of L. Then the set inclusion $D \cap U \to U$ is a weak homotopy equivalence.*

Proof. We shall prove the lemma when $L = h^{-1}(0)$ for some submersion $h\colon V \to \mathbf{R}^n$, where V is an open subset of $\mathbf{E}$. This is sufficient for our purposes and the general assertion can be deduced from this by using a partition of unity subordinate to a suitable cover of L.

Let T be a tubular neighborhood of U in V such that $T \cap L = U$. Let K be a compact simplicial complex and $f\colon K \to U$ a continuous map. We shall obtain a continuous $H\colon [0,2] \times K \to U$ such that $H(0,a) = f(a)$ for every $a \in K$ and $H(\{2\} \times K) \subset D \cap U$. Let e_j denote the j-th vector in

[†]This lemma and a weaker version of (1.13) have already appeared in [10].

the canonical basis for $\mathbf{R}^n$, $e_0 = -\sum_{j=1}^n e_j$ and let $\Delta \subset \mathbf{R}^n$ denote the n-simplex $[e_0, \dots, e_n]$. Let $[x_0, x_1, \dots, x_n] \subset T$ be an n-simplex and $\varphi\colon \Delta \to [x_0, x_1, \dots, x_n]$ be given by

$$\varphi\Big(\sum_{j=0}^n s_j e_j\Big) = \sum_{j=0}^n s_j x_j, \quad \text{where } \sum_{j=0}^n s_j = 1 \text{ and } s_j \geq 0 \text{ for all } j = 0, \dots, n.$$

We shall say that $[x_0, x_1, \dots, x_n]$ is *neat* if $h \circ \varphi\colon \Delta \to \mathbf{R}^n$ is an embedding and $0 \in (h \circ \varphi)(\operatorname{Int} \Delta)$.

Given $p \in T$, let dh_p denote the derivative of h at p and $N_p = \ker(dh_p)$. Define $w_j\colon T \to \mathbf{E}$ by:

$$w_j(p) = \big(dh_p|_{N_p^\perp}\big)^{-1}(e_j) \quad (p \in T,\ j = 0, \dots, n). \tag{7}$$

Notice that $h\big(p + \sum_j \lambda_j w_j(p)\big) = h(p) + \sum_j \lambda_j e_j + o(|\lambda|)$ (for $\lambda = (\lambda_0, \dots, \lambda_n)$ and $p \in T$). Hence, using compactness of K, we can find $r, \varepsilon > 0$ such that:

(i) For any $p \in f(K)$, $[p + rw_0(p), \dots, p + rw_n(p)] \subset T$ and it is neat;
(ii) If $p \in f(K)$ and $|q_j - (p + rw_j(p))| < \varepsilon$ for each j, then $[q_0, \dots, q_n] \subset T$ and it is neat.

Let a_i $(i = 1, \dots, m)$ be the vertices of the triangulation of K. Set $v_i = f(a_i)$ and

$$v_{ij} = v_i + rw_j(v_i) \quad (i = 1, \dots, m,\ j = 0, \dots, n).$$

For each such i, j, choose $\tilde{v}_{ij} \in D \cap T$ with $|\tilde{v}_{ij} - v_{ij}| < \frac{\varepsilon}{2}$. Let

$$v_{ij}(s) = (2 - s)v_{ij} + (s - 1)\tilde{v}_{ij}, \quad \text{so that}$$

$$|v_{ij}(s) - v_{ij}| < \frac{\varepsilon}{2} \quad (s \in [1, 2],\ i = 1, \dots, m,\ j = 0, \dots, n). \tag{8}$$

For any $i, i' \in \{1, \dots, m\}$ and $j = 0, \dots, n$, we have

$$|v_{ij} - v_{i'j}| \leq |f(a_i) - f(a_{i'})| + r\,|w_j \circ f(a_i) - w_j \circ f(a_{i'})|\,.$$

Since f and the w_j are continuous functions, we can suppose that the triangulation of K is so fine that $|v_{ij} - v_{i'j}| < \frac{\varepsilon}{2}$ for each $j = 0, \dots, n$ whenever there exists a simplex having a_i, $a_{i'}$ as two of its vertices. Let $a \in K$ lie in some d-simplex of this triangulation, say, $a = \sum_{i=1}^{d+1} t_i a_i$ (where each $t_i > 0$ and $\sum_i t_i = 1$). Set

$$z_j(s) = \sum_{i=1}^{d+1} t_i v_{ij}(s) \quad (s \in [1, 2],\ j = 0, \dots, n).$$

Then $[z_0(s), \dots, z_n(s)]$ is a neat simplex because condition (ii) is satisfied (with $p = v_1$):

$$\Big|\sum_{i=1}^{d+1} t_i v_{ij}(s) - v_{1j}\Big| \leq \sum_{i=1}^{d+1} t_i\big(\,|v_{ij}(s) - v_{ij}| + |v_{ij} - v_{1j}|\,\big) < \varepsilon,$$

the strict inequality coming from (8) and our hypothesis on the triangulation. Define $H(s, a)$ as the unique element of $h^{-1}(0) \cap [z_0(s), \dots, z_n(s)]$ $(s \in [1, 2])$. Observe that for any $a \in K$, $H(s, a) \in U = h^{-1}(0) \cap T$ $(s \in [1, 2])$ and $H(2, a) \in D \cap U$, as it is the convex combination of the $\tilde{v}_{ij} \in D$.

By reducing $r, \varepsilon > 0$ (and refining the triangulation of K) if necessary, we can ensure that

$$(1 - s)f(a) + sH(1, a) \in T \quad \text{for all } s \in [0, 1] \text{ and } a \in K.$$

Let $\operatorname{pr}\colon T \to U$ be the associated retraction. Complete the definition of H by setting:

$$H(s, a) = \operatorname{pr}\big((1 - s)f(a) + sH(1, a)\big) \quad (s \in [0, 1],\ a \in K).$$

The existence of H shows that f is homotopic within U to a map whose image is contained in $D \cap U$. Taking $K = \mathbf{S}^k$, we conclude that the set inclusion $D \cap U \to U$ induces surjective maps $\pi_k(D \cap U) \to \pi_k(U)$ for all $k \in \mathbf{N}$.

We now establish that the inclusion $D \cap U \to U$ induces injections on all homotopy groups. Let $k \in \mathbf{N}$, $G\colon \mathbf{D}^{k+1} \to U$ be continuous and suppose that the image of $g = G|_{\mathbf{S}^k}$ is contained in $D \cap U$. Let $G_0\colon \mathbf{D}^{k+1} \to D \cap U$ be a close approximation to G; the existence of G_0 was proved above. Let $\varepsilon \in (0, 1)$ and define

$$G_1\colon \mathbf{D}^{k+1} \to D \cap T \text{ by } G_1(a) = \begin{cases} (1 - s)g\big(\frac{a}{|a|}\big) + sG_0\big(\frac{a}{|a|}\big) & \text{if } |a| = (1 - s\varepsilon),\ s \in [0, 1] \\ G_0\big(\frac{a}{1 - \varepsilon}\big) & \text{if } |a| \leq 1 - \varepsilon \end{cases}$$

Notice that we can make G_1 as close as desired to G by a suitable choice of G_0 and ε. Let w_j be as in (7). We claim that there exist continuous functions $\tilde{w}_j \colon \mathbf{D}^{k+1} \to D$ $(j = 0, \dots, n)$ such that:

(I) $\sum_{j=0}^{n} \tilde{w}_j(a) = 0$ for all $a \in \mathbf{D}^{k+1}$;

(II) For any $a \in \mathbf{D}^{k+1}$, $[G_1(a) + \tilde{w}_0(a), \dots, G_1(a) + \tilde{w}_n(a)] \subset D \cap T$ and it is neat.

To prove this, invoke condition (ii) above (with $\mathbf{D}^{k+1}$ in place of K and G in place of f) together with denseness of D to find constant $\tilde{w}_j$ on open sets which cover $\mathbf{D}^{k+1}$, and use a partition of unity. By (II), for each $a \in \mathbf{D}^{k+1}$ there exist unique $t_0(a), \dots, t_n(a) \in [0,1]$ such that $\sum_i t_i(a) = 1$ and

$$G_2(a) = G_1(a) + t_0(a)\tilde{w}_0(a) + \cdots + t_n(a)\tilde{w}_n(a) \in h^{-1}(0).$$

We obtain thus a continuous map $G_2 \colon \mathbf{D}^{k+1} \to D \cap U$. Since $G_1|_{\mathbf{S}^k} = g$ and $h \circ g = 0$, we conclude from (I) and uniqueness of the t_i that $G_2|_{\mathbf{S}^k} = g$. Therefore, G_2 is a nullhomotopy of g in $D \cap U$. □

(1.14) Corollary. *Let $\kappa_1 < \kappa_2$, S be a complete oriented surface and $u, v \in UTS$. Then the subset of all smooth curves in $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ is dense in the latter.*

Proof. Take $\mathbf{E} = L^2[0,1] \times L^2[0,1]$, $D = C^\infty[0,1] \times C^\infty[0,1]$ and U an open subset of $L = \mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$. Then it is a trivial consequence of (1.13) that $D \cap U \neq \emptyset$ if $U \neq \emptyset$. □

(1.15) Corollary. *Let $\kappa_1 < \kappa_2$, S be a complete oriented surface, $u, v \in UTS$ and $\mathcal{U} \subset \mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ be open. Let K be a compact simplicial complex and $f \colon K \to \mathcal{U}$ be a continuous map. Then there exists a continuous $g \colon K \to \mathcal{U}$ such that:*

(i) *$f \simeq g$ within $\mathcal{U}$.*

(ii) *$g(a)$ is a smooth curve for all $a \in K$.*

(iii) *All derivatives of $g(a)$ with respect to t depend continuously on $a \in K$.*

In particular, the map $j \colon \mathcal{C}^{\kappa_2}_{\kappa_1}(u, v) \to \mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ given by (6) induces surjections $\pi_k(j^{-1}(\mathcal{U})) \to \pi_k(\mathcal{U})$ for all $k \in \mathbf{N}$.

Proof. Parts (i) and (ii) are exactly what was established in the first part of the proof of (1.13), in the special case where $\mathbf{E} = L^2[0,1] \times L^2[0,1]$, $D = C^\infty[0,1] \times C^\infty[0,1]$, $L = \mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ and $U = \mathcal{U}$. The image of the function $g = H_2 \colon K \to \mathcal{U}$ constructed there is contained in a finite-dimensional vector subspace of D, namely, the one generated by all $\tilde{v}_{ij}$, so (iii) also holds. □

(1.16) Lemma. *Let $\kappa_1 < \kappa_2$, S be a complete oriented surface and $u, v \in UTS$. Then the inclusion $i \colon \mathcal{C}^{\kappa_2}_{\kappa_1}(u, v)^r \to \mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ is a homotopy equivalence for any $r \in \mathbf{N}$, $r \geq 2$. Consequently, $\mathcal{C}^{\kappa_2}_{\kappa_1}(u, v)^r$ is homeomorphic to $\mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ for any $r \in \mathbf{N}$, $r \geq 2$.*

Proof. Let $\mathbf{E} = L^2[0,1] \times L^2[0,1]$, let $\mathbf{F} = C^{r-1}[0,1] \times C^{r-2}[0,1]$ (where $C^k[0,1]$ denotes the set of all C^k functions $[0,1] \to \mathbf{R}$, with the C^k norm) and let $i \colon \mathbf{F} \to \mathbf{E}$ be set inclusion. Setting $\mathcal{M} = \mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$, we conclude from (1.12(c)) that $i \colon \mathcal{N} = i^{-1}(\mathcal{M}) \hookrightarrow \mathcal{M}$ is a homotopy equivalence. We claim that $\mathcal{N}$ is homeomorphic to $\mathcal{C}^{\kappa_2}_{\kappa_1}(u, v)^r$, where the homeomorphism is obtained by associating a pair $(\hat{\sigma}, \hat{\kappa}) \in \mathcal{N}$ to the curve γ obtained by solving (3), with σ and κ as in (4).

Suppose first that $\gamma \in \mathcal{C}^{\kappa_2}_{\kappa_1}(u, v)^r$. Then $|\dot{\gamma}|$ (resp. κ) is a function $[0,1] \to \mathbf{R}$ of class C^{r-1} (resp. C^{r-2}). Hence, so are $\hat{\sigma} = h \circ |\dot{\gamma}|$ and $\hat{\kappa} = h^{\kappa_2}_{\kappa_1} \circ \kappa$, since h and $h^{\kappa_2}_{\kappa_1}$ are smooth. Moreover, if γ, $\eta \in \mathcal{C}^{\kappa_2}_{\kappa_1}(u, v)^r$ are close in C^r topology, then $\hat{\kappa}_\gamma$ is C^{r-2}-close to $\hat{\kappa}_\eta$ and $\hat{\sigma}_\gamma$ is C^{r-1}-close to $\hat{\sigma}_\eta$.

Conversely, if $(\hat{\sigma}, \hat{\kappa}) \in \mathcal{N}$, then $\sigma = h^{-1} \circ \hat{\sigma}$ is of class C^{r-1} and $\kappa = (h^{\kappa_2}_{\kappa_1})^{-1} \circ \hat{\kappa}$ of class C^{r-2}. Since all functions on the right side of (3) are of class (at least) C^{r-2}, the solution $\mathbf{t} = \mathbf{t}_\gamma$ to this initial value problem is of class C^{r-1}. Moreover, $\dot{\gamma} = \sigma\mathbf{t}$, hence the velocity vector of γ is seen to be of class C^{r-1}. We conclude that γ is a curve of class C^r. Further, continuous dependence on the parameters of a differential equation shows that the correspondence $(\hat{\sigma}, \hat{\kappa}) \mapsto \mathbf{t}_\gamma$ is continuous. Since γ is obtained by integrating $\sigma\mathbf{t}_\gamma$, we deduce that the map $(\hat{\sigma}, \hat{\kappa}) \mapsto \gamma$ is likewise continuous.

The last assertion of the lemma follows from (1.12(b)). □

The next two results allow us to reparametrize a family of curves to better suit our needs.

(1.17) Lemma. *Let $\mathcal{M} = \mathcal{L}^{\kappa_2}_{\kappa_1}(u, v)$ or $\mathcal{M} = \mathcal{C}^{\kappa_2}_{\kappa_1}(u, v)$. Let A be a topological space and $A \to \mathcal{M}$, $a \mapsto \gamma_a$, be a continuous map. Then there exists a homotopy $\gamma^r_a \colon [0,1] \to \mathcal{M}$, $r \in [0,1]$, such that for any $a \in A$:*

(i) $\gamma_a^0 = \gamma_a$ *and* γ_a^1 *is parametrized so that* $|\dot\gamma_a^1(t)|$ *is independent of* t.
(ii) γ_a^r *is an orientation-preserving reparametrization of* γ_a, *for all* $r \in [0,1]$.

Proof. Let $s_a(t) = \int_0^t |\dot\gamma_a(\tau)|\, d\tau$ be the arc-length parameter of γ_a, L_a its length and $\tau_a\colon [0, L_a] \to [0,1]$ the inverse function of s_a. Define $\gamma_a^r\colon [0,1] \to M$ by:

$$\gamma_a^r(t) = \gamma_a\big((1-r)t + r\tau_a(L_a t)\big) \qquad (r, t \in [0,1],\ a \in A).$$

Then γ_a^r is the desired homotopy. □

(1.18) Corollary. *Let* $\mathcal{M} = \mathcal{L}^{\kappa_2}_{\kappa_1}(u,v)$ *or* $\mathcal{M} = \mathcal{C}^{\kappa_2}_{\kappa_1}(u,v)$. *Let* A *be a topological space and* $f\colon \mathbf{S}^0 \times A \to \mathcal{M}$ *a continuous map such that, for all* $a \in A$, $f(1,a)$ *is an orientation-preserving reparametrization of* $f(-1,a)$. *Then* f *admits a continuous extension* $F\colon A \times [-1,1] \to \mathcal{M}$ *with the property that* $f(r,a)$ *is an orientation-preserving reparametrization of* $f(-1,a)$ *for all* $r \in [-1,1]$, $a \in A$. □

It is assumed in (1.17) and (1.18) that the parametrizations of all curves have domain $[0,1]$, but it is clearly possible to have these intervals depend (continuously) on a. A typical application is to reparametrize all curves in a family by arc-length (not just proportionally to arc-length as in (1.17)).

2. Plane curves

Basic terminology. As mentioned before, we shall identify $\mathbf{R}^2$ with $\mathbf{C}$ throughout. As a set, the unit tangent bundle $UT\mathbf{C}$ may then be regarded as $\mathbf{C} \times \mathbf{S}^1$. We define the *frame* Φ_γ of an admissible curve $\gamma\colon [0,1] \to \mathbf{C}$ to be the lift of γ to $UT\mathbf{C} \equiv \mathbf{C} \times \mathbf{S}^1$:

$$\Phi_\gamma\colon [0,1] \to \mathbf{C} \times \mathbf{S}^1, \quad \Phi_\gamma(t) = (\gamma(t), \mathbf{t}(t)). \tag{9}$$

Here $\mathbf{t} = \mathbf{t}_\gamma\colon [0,1] \to \mathbf{S}^1$ is the *unit tangent* to γ.[†] The *unit normal* $\mathbf{n} = \mathbf{n}_\gamma\colon [0,1] \to \mathbf{S}^1$ is given by $\mathbf{n} = i\mathbf{t}$ (where i denotes the imaginary unit). An *argument* $\theta_\gamma\colon [0,1] \to \mathbf{R}$ of $\mathbf{t}$ is a continuous function satisfying $e^{i\theta_\gamma(t)} = \mathbf{t}(t)$ for every $t \in [0,1]$. The *total turning* of γ is defined to be $\theta_\gamma(L) - \theta_\gamma(0)$; notice that this is independent of the choice of $\theta_\gamma(0)$.

(2.1) Definition. Let $P = (p,w), Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$. From now on we shall denote by $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ the space described in (1.7) when $S = \mathbf{C}$ and P, Q, Φ_γ take the place of $u, v, \mathbf{t}_\gamma$. Given $\theta_1 \in \mathbf{R}$ satisfying $e^{i\theta_1} = z\bar w$, we denote by $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$ the subspace of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ consisting of all curves which have total turning equal to θ_1. When $P = (0,1)$, the space $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ (resp. $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$) will be denoted simply by $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q)$ (resp. $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q;\theta_1)$).

It should be intuitively clear that $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ is never empty. In contrast, $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$ may be empty if $\kappa_1\kappa_2 \geq 0$, depending on the value of θ_1. The proof of these assertions is postponed to §5. Notice that $\Phi_\gamma, \mathbf{t}_\gamma, \mathbf{n}_\gamma$ and θ_γ are defined *everywhere* on $[0,1]$ for an admissible curve γ, even if it is not of class C^2. In fact, they are all absolutely continuous.

(2.2) Lemma. *Let* $\gamma \in \mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$, $\mathbf{t}\colon [0,1] \to \mathbf{S}^1$ *be its unit tangent and* $\theta\colon [0,1] \to \mathbf{R}$ *an argument of* $\mathbf{t}$. *Then* $\mathbf{t}$ *and* θ *are absolutely continuous functions.*

Proof. Since $\theta = \arg \circ \mathbf{t}$, the absolutely continuity of θ is equivalent to that of $\mathbf{t}$. By definition, $\mathbf{t}$ is obtained as the solution of the initial value problem (3), where κ, σ lie in $L^2[0,1]$. Since $L^2[0,1] \subset L^1[0,1]$, it follows from thm. C.3 on p. 386 of [13] that $\mathbf{t}$ is absolutely continuous (cf. also p. 385 of the same book). □

Let $\gamma\colon [0,L] \to \mathbf{C}$ be an admissible curve, parametrized by arc-length. Its *curvature* $\kappa_\gamma = \kappa$ (which is defined only a.e.) is given by

$$\kappa(s) = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle \quad (s \in [0,L]). \tag{10}$$

In terms of a more general parameter t,

$$\kappa(t) = \frac{1}{|\dot\gamma(t)|} \left\langle \dot{\mathbf{t}}(t), \mathbf{n}(t) \right\rangle.$$

[†]Strictly speaking, this definition of $\mathbf{t}_\gamma$ does not agree with the one in §1; the analogue of $\mathbf{t}_\gamma$ when $S = \mathbf{C}$ is really Φ_γ. Since we will henceforth work exclusively with plane curves except in §9, this should not lead to confusion.

[We denote derivatives with respect to arc-length by a $'$ (prime) and derivatives with respect to other parameters by a $\dot{}$ (dot).] If γ is twice differentiable, then we also have

$$\kappa(t) = \frac{\det(\dot\gamma(t), \ddot\gamma(t))}{|\dot\gamma(t)|^3}. \tag{11}$$

It follows from (10) that if $\theta_\gamma\colon [0,L] \to \mathbf{R}$ is an argument of $\mathbf{t}$, then

$$\kappa(s) = \theta_\gamma'(s). \tag{12}$$

The *radius of curvature* ρ of γ is given by $\rho = \frac{1}{\kappa}$; when $\kappa(t) = 0$, it is to be understood that $\rho(t) = \infty$ (unsigned infinity).

The group structure of $UT\mathbf{C}$. An orientation-preserving isometry of $\mathbf{C}$ (i.e., a proper Euclidean motion) is uniquely determined by where it maps $(0,1) \in UT\mathbf{C}$. Thus, $UT\mathbf{C}$ also carries a natural Lie group structure as a semidirect product $\mathbf{C} \rtimes \mathbf{S}^1$, wherein the operation is given by

$$(p,w)\cdot(q,z) = (p+wq, wz) \qquad (p,q\in\mathbf{C},\ w,z\in\mathbf{S}^1).$$

If we identify its Lie algebra with $\mathbf{C}\times\mathbf{R}$, then the bracket operation is given by

$$[(a,\theta),(b,\varphi)] = \big(i(\theta b - \varphi a), 0\big) \qquad (a,b\in\mathbf{C},\ \theta,\varphi\in\mathbf{R}). \tag{13}$$

Viewed as a family of Euclidean motions, the frame Φ_γ of an admissible curve operates on $\mathbf{C}$ through

$$\Phi_\gamma(t)a = \gamma(t) + \mathbf{t}(t)a \qquad (a\in\mathbf{C},\ t\in[0,1]). \tag{14}$$

We can also realize $UT\mathbf{C}$ as a matrix group if we identify

$$P = (p,w) \text{ with } \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } p = x+iy,\ w = e^{i\theta}.$$

The expression for the bracket in (13) can be derived from this. Under this identification,

$$\Phi_\gamma\colon [0,1]\to\mathbf{GL}_3, \quad \Phi_\gamma(t) = \begin{pmatrix} \cos\theta_\gamma(t) & -\sin\theta_\gamma(t) & \gamma_1(t) \\ \sin\theta_\gamma(t) & \cos\theta_\gamma(t) & \gamma_2(t) \\ 0 & 0 & 1 \end{pmatrix},$$

where $\theta_\gamma\colon[0,1]\to\mathbf{R}$ is an argument of $\mathbf{t}_\gamma$ and $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$.[†] Using (12), we deduce that

$$\dot\Phi_\gamma = |\dot\gamma| \begin{pmatrix} -\kappa_\gamma\sin\theta_\gamma & -\kappa_\gamma\cos\theta_\gamma & \cos\theta_\gamma \\ \kappa_\gamma\cos\theta_\gamma & -\kappa_\gamma\sin\theta_\gamma & \sin\theta_\gamma \\ 0 & 0 & 0 \end{pmatrix} = \Phi_\gamma\Lambda_\gamma, \text{ where } \Lambda_\gamma = |\dot\gamma| \begin{pmatrix} 0 & -\kappa_\gamma & 1 \\ \kappa_\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{a}\subset\mathfrak{gl}_3$ denote the Lie subalgebra of $UT\mathbf{C}$, which is generated by

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

We call the map $\Lambda_\gamma\colon[0,1]\to\mathfrak{a}$ the *logarithmic derivative* of γ; note that the entries of Λ_γ belong to $L^2[0,1]$, so that they are defined only a.e.. A more important observation is that not every solution $\Phi\colon[0,1]\to\mathbf{GL}_3$ of the differential equation $\dot\Phi = \Phi\Lambda$ for some $\Lambda\colon[0,1]\to\mathfrak{a}$ is the frame of an admissible curve γ. This occurs if and only if the image of Λ is contained in the half-plane

$$\mathfrak{h} = \big\{aA + bB : a\in\mathbf{R},\ b>0\big\} \subset \mathfrak{a}. \tag{16}$$

If this is the case, then $\kappa_\gamma(t)$ is given as the quotient of $\Lambda^{(2,1)}(t)$ by $\Lambda^{(1,3)}(t)$, where $\Lambda^{(i,j)}$ denotes the (i,j)-entry of Λ. Hence, if $\kappa_1<\kappa_2$ and we define $\mathfrak{k}\subset\mathfrak{h}$ to be the convex cone

$$\mathfrak{k} = \big\{aA+bB : a\in\mathbf{R},\ b>0 \text{ and } \kappa_1 b < a < \kappa_2 b\big\},$$

then the curvature of γ takes values in (κ_1,κ_2) if and only if the image of Λ is contained in $\mathfrak{k}$.

[†]Notice that the first column of Φ_γ gives the coordinates of $\mathbf{t}_\gamma$, the second the coordinates of $\mathbf{n}_\gamma$ and the third the coordinates of γ. This justifies our terminology "frame" for Φ_γ.

This yields an alternative definition of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,\cdot)$ (when $S=\mathbf{C}$). Let $\gamma\colon[0,1]\to\mathbf{C}$, $\gamma=\gamma_1+i\gamma_2$. We say that $\gamma\in\mathcal{L}^{\kappa_2}_{\kappa_1}(P,\cdot)$ if $\gamma_1=\Phi^{(1,3)}$ and $\gamma_2=\Phi^{(2,3)}$ for some solution $\Phi\colon[0,1]\to\mathbf{GL}_3$ of the initial value problem

$$\Phi(0)=P,\quad \dot\Phi=\Phi\Lambda,\quad \text{where}\ \ \Lambda\colon[0,1]\to\mathfrak{k},\quad \Lambda(t)=\sigma(t)\begin{pmatrix}0&-\kappa(t)&1\\ \kappa(t)&0&0\\ 0&0&0\end{pmatrix}$$

and σ, κ are as in (4), with $(\hat\sigma,\hat\kappa)\in\mathbf{E}=L^2[0,1]\times L^2[0,1]$. Using thm. C.3 in [13], one concludes that Φ is uniquely determined by the pair $(\hat\sigma,\hat\kappa)$; the set $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,\cdot)$ inherits a topology from $\mathbf{E}$ via the identification $\gamma\leftrightarrow(\hat\sigma,\hat\kappa)$. This definition is clearly equivalent to that in (1.5), since all that we have done is to reformulate eq. (3) in another language.

Remark. There is no natural Lie group structure on UTS when S is an arbitrary surface. However, if $S=\mathbf{R}^2$, $\mathbf{S}^2$ or $\mathbf{H}^2$, then the group consisting of all orientation-preserving isometries of S acts simply transitively on UTS, allowing us to identify the latter with this group. The definition of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,\cdot)$ presented here is analogous to the one in [8] and [10], which study the topology of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ when $S=\mathbf{S}^2$.

Proof of (1.6). Assume first that $S=\mathbf{R}^2$ with the Euclidean metric. Let $\delta>0$, $r\in(-\delta,\delta)$ and $\hat\sigma(r)$, $\hat\kappa(r)\in L^2[0,1]$ be one-parameter families of functions; set $\sigma(r)=h^{-1}\circ\hat\sigma(r)$, $\kappa(r)=(h^{\kappa_2}_{\kappa_1})^{-1}\circ\hat\kappa(r)$. Define a corresponding family of curves $\Lambda(r)\colon[0,1]\to\mathfrak{k}$ by

$$\Lambda(r)=\sigma(r)\begin{pmatrix}0&-\kappa(r)&1\\ \kappa(r)&0&0\\ 0&0&0\end{pmatrix}.$$

Denoting derivatives with respect to t (resp. r) by a $\dot{}$ (resp. $'$), let $\Phi(r)\colon[0,1]\to UT\mathbf{C}$, $t\mapsto\Phi(r)(t)$, be the solution of $\dot\Phi(r)=\Phi(r)\Lambda(r)$. A straightforward computation shows that

$$\Phi'(r)(t)\big[\Phi(r)(t)\big]^{-1}=\int_0^t\Phi(r)(\tau)\,\Lambda'(r)(\tau)\,\big[\Phi(r)(\tau)\big]^{-1}\,d\tau\quad(r\in(-\delta,\delta),\ t\in[0,1]).$$

Let $\Lambda'(0)$ consist of three smooth narrow bumps at times $t=t_0$, $t=t_1$ and $t=t_2$, for $t_i\in(0,1)$ close to 1. Let $\Psi=\Phi(0)$; setting $r=0$ in the previous expression, we deduce that

$$\Psi(1)^{-1}\Phi'(0)(1)\approx\sum_{i=1}^3\big[\Psi(t_i)^{-1}\Psi(1)\big]^{-1}\Lambda'(0)(t_i)\,\big[\Psi(t_i)^{-1}\Psi(1)\big].$$

Since each $\Lambda(r)$ is a curve in $\mathfrak{k}$, we can make $\Lambda'(0)(t_i)$ assume any value in the vector subspace $\mathfrak{v}$ generated by A and B (as in (15)). Another computation using the fact that $\sigma(0)>0$ a.e. shows that the planes $\mathfrak{v}$ and $\big[\Psi(t_i)^{-1}\Psi(1)\big]^{-1}\mathfrak{v}\big[\Psi(t_i)^{-1}\Psi(1)\big]$ are transversal for small $1-t_i$, with the angle between them proportional to $(1-t_i)+o(1-t_i)$. Hence, any vector in $\mathfrak{a}$ can be written in the form $\Psi(1)^{-1}\Phi'(0)(1)$ for a suitable choice of $\Lambda'(0)$, which shows that F is a submersion when $S=\mathbf{R}^2$.

Suppose now that S is an arbitrary complete oriented surface, and $\gamma\in\mathcal{L}^{\kappa_2}_{\kappa_1}(u,v)$ for some $u,v\in UTS$. We can use Riemannian normal coordinates centered at $\gamma(1)$ to obtain a chart x for which the metric satisfies

$$g_{ij}(x)=\delta_{ij}+\sum_{k,l=1}^2\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}(0)x^kx^l+o(|x|^2),$$

where δ_{ij} is Kronecker delta. It follows from what was proved above for $\mathbf{R}^2$ with the Euclidean metric that the derivative of F at γ is surjective. □

Spaces of curves with curvature in a closed interval.

(2.3) Definition. Let $P,Q\in UT\mathbf{C}$ and $-\infty<\kappa_1<\kappa_2<+\infty$. Define $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$ to be the set of all C^1 regular plane curves $\gamma\colon[0,1]\to\mathbf{C}$ satisfying:

(i) $\Phi_\gamma(0)=P$ and $\Phi_\gamma(1)=Q$;

(ii) $\kappa_1\le\frac{\theta(s_1)-\theta(s_2)}{s_1-s_2}\le\kappa_2$ for any $s_1\neq s_2\in[0,L]$. (Here the parameter is the arc-length of γ, L is its length and $\theta\colon[0,L]\to\mathbf{R}$ an argument of $\mathbf{t}_\gamma$.)

Condition (ii) implies that θ is a Lipschitz function. In particular, it is absolutely continuous, and its derivative κ_γ lies in L^2, since it is bounded. We give this set the topology induced by the following distance function d: Given $\gamma, \eta \in \hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$, set

$$d(\gamma,\eta) = \|\gamma-\eta\|_2 + \|\dot\gamma-\dot\eta\|_2 + \|\kappa_\gamma - \kappa_\eta\|_2 .$$

For $P=(0,1)\in \mathbf{C}\times\mathbf{S}^1$, we will denote $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$ simply by $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(Q)$.

Remark. This definition is essentially due to L. E. Dubins, who studied paths of minimal length, now called *Dubins paths*, in $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(P,Q)$ ($\kappa_0>0$). Such shortest paths always exist, but may not be unique in some special cases (see prop. 1 and the corollary to thm. I of [2]). His main result states that any Dubins path is the concatenation of at most three pieces, each of which is either a line segment or an arc of circle of radius $\frac{1}{\kappa_0}$ (see thm. I of [2] for the precise statement). Dubins paths and variations thereof have many applications in engineering and are the subject of a vast literature, which we are not qualified to survey. The space $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$ will play a minor role in our investigations. Its topology has been chosen to ensure that the following result holds.

(2.4) Lemma. *Let $(\kappa_1,\kappa_2)\subset[\bar\kappa_1,\bar\kappa_2]\subset(\bar{\bar\kappa}_1,\bar{\bar\kappa}_2)$ and $P,Q\in UT\mathbf{C}$. Then the set inclusions $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)\to\hat{\mathcal{L}}^{\bar\kappa_2}_{\bar\kappa_1}(P,Q)$ and $\hat{\mathcal{L}}^{\bar\kappa_2}_{\bar\kappa_1}(P,Q)\to\mathcal{L}^{\bar{\bar\kappa}_2}_{\bar{\bar\kappa}_1}(P,Q)$ are continuous injections.*

Proof. The proof is a straightforward verification, which will be left to the reader. □

3. Normal translation

An analogue of the following construction has already appeared in [10]. It can be used to uniformly shift the radii of curvature of a family of curves.

(3.1) Definition. Let $\gamma\colon[0,1]\to\mathbf{C}$ be admissible and $u\in\mathbf{R}$. The *normal translation* γ_u of γ by u is the curve given by

$$\gamma_u(t) = \gamma(t) + u\mathbf{n}(t) \quad (t\in[0,1]).$$

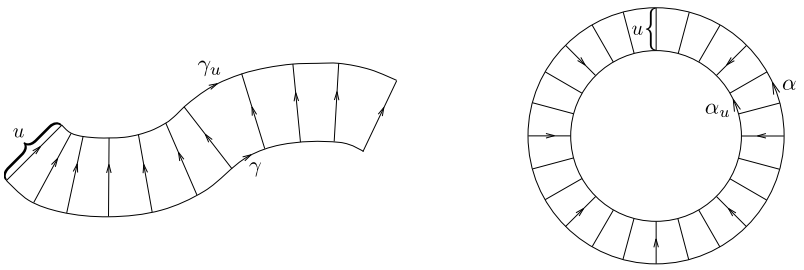


FIGURE 3. The normal translation of a general curve γ and of a circle α.

Observe that the normal translation α_u of a circle α of radius of curvature $\rho\in\mathbf{R}$ ($\rho\neq0$) is a circle of radius of curvature $\rho-u$ for any u in the component of $\mathbf{R}\smallsetminus\{\rho\}$ containing 0 (see fig. 3). The following lemma generalizes this to arbitrary curves.

(3.2) Lemma. *Let $\gamma\colon[0,1]\to\mathbf{C}$ be admissible and κ, ρ, $\mathbf{t}$ denote its curvature, radius of curvature and unit tangent, respectively. Let $u\in\mathbf{R}$ be such that $1-u\kappa(t)>0$ for almost every $t\in[0,1]$. Then the normal translation γ_u of γ by u has the following properties:*

(a) *γ_u is admissible and its unit tangent vector $\hat{\mathbf{t}}$ satisfies $\hat{\mathbf{t}}(t)=\mathbf{t}(t)$ for each $t\in[0,1]$. In particular, γ and γ_u have the same total turning.*
(b) *$(\gamma_u)_{-u}=\gamma$.*
(c) *For almost every $t\in[0,1]$, the curvature $\hat\kappa$ of γ_u is given by:*

$$\hat\kappa(t) = \frac{\kappa(t)}{1-u\kappa(t)}$$

and its radius of curvature $\hat\rho$ by:

$$\hat\rho(t) = \rho(t) - u.^\dagger$$

(d) *Let $\Phi_\gamma(0) = (p, w)$, $\Phi_\gamma(1) = (q, z) \in \mathbf{C} \times \mathbf{S}^1$. Then*

$$\Phi_{\gamma_u}(0) = (p + iuw, w), \quad \Phi_{\gamma_u}(1) = (q + iuz, z).$$

Proof. Let $\theta_\gamma\colon [0,1] \to \mathbf{R}$ be an argument of $\mathbf{t}$ and define $\Psi\colon [0,1] \to \mathbf{GL}_3$ by

$$\Psi = \begin{pmatrix} \cos\theta_\gamma & -\sin\theta_\gamma & \gamma_1 - u\sin\theta_\gamma \\ \sin\theta_\gamma & \cos\theta_\gamma & \gamma_2 + u\cos\theta_\gamma \\ 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

A straightforward calculation shows that Ψ satisfies $\dot\Psi = \Psi\Lambda$, for

$$\Lambda\colon [0,1] \to \mathfrak{a} \subset \mathfrak{gl}_3, \quad \Lambda = \begin{pmatrix} 0 & -\kappa & 1 - u\kappa \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By hypothesis, the image of Λ is contained in the half-plane $\mathfrak{h}$ of (16). Comparing the third column of (17) with the definition of γ_u, we deduce that γ_u is admissible. Further, looking at the first and second columns we deduce that $\hat{\mathbf{t}} = \mathbf{t}$ and $\hat{\mathbf{n}} = \mathbf{n}$. Consequently,

$$(\gamma_u)_{-u} = \gamma_u - u\hat{\mathbf{n}} = (\gamma + u\mathbf{n}) - u\mathbf{n} = \gamma.$$

This establishes (a) and (b). We know that the curvature $\hat\kappa$ of γ_u is given by the quotient of $\Lambda^{(2,1)}$ by $\Lambda^{(1,3)}$, that is,

$$\hat\kappa = \frac{\kappa}{1 - u\kappa} = \frac{1}{\rho - u} = \frac{1}{\hat\rho}. \qquad \square$$

This proves (c). Part (d) is an immediate consequence of the definition and of (a).

The next result greatly simplifies the study of the spaces $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$.

(3.3) Theorem A. *Let $P = (p, w)$, $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$, $-\infty \le \kappa_1 < \kappa_2 \le +\infty$ and $\rho_i = \frac{1}{\kappa_i}$.*

(a) *Suppose $\kappa_1 < 0 < \kappa_2$. If at least one of κ_1, κ_2 is finite, then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q) \approx \mathcal{L}^{+1}_{-1}(Q_1)$ for*

$$Q_1 = \Big(\tfrac{2}{\rho_2 - \rho_1}\bar w\big[(q-p) + \tfrac{i}{2}(\rho_1 + \rho_2)(z - w)\big]\,,\, z\bar w\Big).$$

(b) *Suppose $0 < \kappa_1 < \kappa_2$. Then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q) \approx \mathcal{L}^{+\infty}_{1}(Q_2)$ for*

$$Q_2 = \Big(\tfrac{\bar w}{\rho_1 - \rho_2}\big[(q-p) + i\rho_2(z - w)\big]\,,\, z\bar w\Big).$$

(c) *Suppose $0 = \kappa_1 < \kappa_2$. Then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q) \approx \mathcal{L}^{+\infty}_{0}(Q_3)$ for*

$$Q_3 = \big(\bar w\,[(q-p) + i\rho_2(z - w)]\,,\, z\bar w\big).$$

(d) *Suppose $\kappa_1 < \kappa_2 < 0$. Then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q) \approx \mathcal{L}^{+\infty}_{1}(Q_4)$ for*

$$Q_4 = \Big(\tfrac{\bar z}{\rho_1 - \rho_2}\big[(q-p) + i\rho_1(z - w)\big]\,,\, w\bar z\Big).$$

(e) *Suppose $\kappa_1 < \kappa_2 = 0$. Then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q) \approx \mathcal{L}^{+\infty}_{0}(Q_5)$ for*

$$Q_5 = \big(\bar z\,[(q-p) + i\rho_1(z - w)]\,,\, w\bar z\big).$$

In cases (a)–(c) (resp. (d)–(e)), the total turning of the image of a curve under the homeomorphism is equal (resp. opposite) to that of the original curve.

The notation $X \approx Y$ means that X is homeomorphic to Y.[‡]

[†]Recall that we have defined $\rho(t) = \infty$ when $\kappa(t) = 0$; we adopt the convention that $\infty + a = \infty$ for any $a \in \mathbf{R}$.

[‡]Note that (a)–(e) cover every possibility for the position of κ_1, κ_2, except for $\kappa_2 = +\infty = -\kappa_1$. As discussed in the introduction, the corresponding space $\mathcal{L}^{+\infty}_{-\infty}(P, Q)$ has infinitely many connected components, one for each possible argument of $z\bar w$, and these components are all contractible. For this reason, this case is largely ignored.

Proof. Suppose that $\kappa_1 < 0 < \kappa_2$, let $\gamma \in \mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ and $\kappa = \frac{1}{\rho}$ be its curvature. If $\rho_1 + \rho_2 \le 0$, then

$$1 - \left(\tfrac{\rho_1+\rho_2}{2}\right)\kappa(t) > 1 - \left(\tfrac{\rho_1+\rho_2}{2}\right)\kappa_1 = \tfrac{1}{2}\big(1 - \rho_2\kappa_1\big) \ge \tfrac{1}{2} > 0 \quad \text{for a.e. } t \in [0,1]$$

and if $\rho_1 + \rho_2 \ge 0$, then

$$1 - \left(\tfrac{\rho_1+\rho_2}{2}\right)\kappa(t) > 1 - \left(\tfrac{\rho_1+\rho_2}{2}\right)\kappa_2 = \tfrac{1}{2}\big(1 - \rho_1\kappa_2\big) \ge \tfrac{1}{2} > 0 \quad \text{for a.e. } t \in [0,1].$$

In terms of the radius of curvature of γ, the hypotheses $\kappa_1 < \kappa < \kappa_2$ and $\kappa_1 < 0 < \kappa_2$ mean that

$$\rho \in (-\infty, \rho_1) \cup \{\infty\} \cup (\rho_2, +\infty).$$

Let $\rho_0 = \frac{\rho_2 - \rho_1}{2}$, $\kappa_0 = \frac{1}{\rho_0}$ and $\hat\rho = \frac{1}{\hat\kappa}$ denote the radius of curvature of the normal translation $\gamma_{\frac{\rho_1+\rho_2}{2}}$. Invoking (3.2(c)), we deduce that

$$\hat\rho \in (-\infty, -\rho_0) \cup \{\infty\} \cup (+\rho_0, +\infty),$$

or, equivalently, $-\kappa_0 < \hat\kappa < +\kappa_0$. We have thus established that translation by $\frac{\rho_1+\rho_2}{2}$ is a map

$$F\colon \mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q) \to \mathcal{L}^{+\kappa_0}_{-\kappa_0}(\hat P, \hat Q),$$

where

$$\hat P = \big(p + \tfrac{\rho_1+\rho_2}{2} iw, w\big),\ \hat Q = \big(q + \tfrac{\rho_1+\rho_2}{2} iz, z\big) \text{ and } \kappa_0 = \frac{2}{\rho_2 - \rho_1}.$$

Clearly, this map is continuous. Reversing the argument we find that if $\hat\gamma \in \mathcal{L}^{+\kappa_0}_{-\kappa_0}(\hat P, \hat Q)$, then its normal translation $\hat\gamma_{-\frac{\rho_1+\rho_2}{2}}$ lies in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$. Hence, by (3.2(b)), normal translation by $-\frac{\rho_1+\rho_2}{2}$ is the inverse of F, and F is a homeomorphism.

Let $T\colon \mathbf{C} \to \mathbf{C}$ be the dilatation $x \mapsto \kappa_0 x$. If $\gamma \in \mathcal{L}^{+\kappa_0}_{-\kappa_0}(\hat P, \hat Q)$, then $T \circ \gamma$ lies in

$$\mathcal{L}^{+1}_{-1}(\tilde P, \tilde Q), \quad \text{where } \tilde P = \big(\kappa_0(p + \tfrac{\rho_1+\rho_2}{2} iw), w\big),\ \tilde Q = \big(\kappa_0(q + \tfrac{\rho_1+\rho_2}{2} iz), z\big)$$

and the correspondence $\gamma \mapsto T \circ \gamma$ yields a homeomorphism between these two spaces. Write $\tilde P = (\tilde p, w) \in \mathbf{C} \times \mathbf{S}^1$ and let $E\colon \mathbf{C} \to \mathbf{C}$ be the Euclidean motion given by $E(x) = \bar w(x - \tilde p)$. Then the map $\gamma \mapsto E \circ \gamma$ is a homeomorphism from $\mathcal{L}^{+1}_{-1}(\tilde P, \tilde Q)$ onto $\mathcal{L}^{+1}_{-1}(Q_1)$, with Q_1 as in the statement.

The proofs of parts (b) and (c) are analogous, so only a brief outline will be provided. In part (b) we first use normal translation by ρ_2, and then compose with the dilatation $x \mapsto \frac{x}{\rho_1 - \rho_2}$ and an Euclidean motion; in part (c) the dilatation is not necessary. Parts (d) and (e) follow from (b) and (c), respectively, by reversing the orientation of all curves in the corresponding space.

By (3.2(a)), the normal translations used in establishing (a)–(c) preserve the total turning of a curve. Clearly, so do dilatations and Euclidean motions, while a reversal of orientation changes the sign of the total turning. This proves the last assertion of the theorem. □

(3.4) *Remark.* Normal translations, and hence also the homeomorphisms constructed in (3.3), do not generally respect inequalities between lengths. This is clear from fig. 3: Two circles of the same radius $r > 0$ but different orientations are mapped to circles of radii equal to $r \pm u$ under normal translation by $u \in (0, r)$. See also the remarks at the end of §10.

A more concise version of (3.3) is the following.[†]

(3.5) Corollary. *Let $P, Q \in UT\mathbf{C}$. Then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ is homeomorphic to a space of type $\mathcal{L}^{+1}_{-1}(Q_0)$, $\mathcal{L}^{+\infty}_{0}(Q_0)$ or $\mathcal{L}^{+\infty}_{1}(Q_0)$, according as $\kappa_1\kappa_2 < 0$, $\kappa_1\kappa_2 = 0$ or $\kappa_1\kappa_2 > 0$, respectively.* □

Out of the three possibilities, the spaces of type $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ with $\kappa_1\kappa_2 < 0$ are the ones with the most interesting topological properties. We deal with the two remaining cases in §8. In the statements of (3.3) and (3.5), we may replace $\mathcal{L}$ with $\mathcal{C}$; the proof of (3.3) goes through unchanged. There is also a version of these results for the spaces of type $\hat{\mathcal{L}}$, see (10.1).

[†]Recall that $0(\pm\infty) = 0$ by convention.

4. Condensed curves

(4.1) Definition. Let $\gamma\colon [0,1] \to \mathbf{C}$ be a regular curve and $\theta\colon [0,1] \to \mathbf{R}$ be an argument of $\mathbf{t}_\gamma$. The *amplitude* of γ is given by

$$\omega = \max_{t\in[0,1]} \theta(t) - \min_{t\in[0,1]} \theta(t).$$

We call γ *condensed*, *critical* or *diffuse* according as $\omega < \pi$, $\omega = \pi$ or $\omega > \pi$.

Our main objective now is to understand the topology of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ when $\kappa_1\kappa_2 < 0$. By (3.5), no generality is lost in assuming that $\kappa_1 = -1$, $\kappa_2 = 1$ and $P = (0,1) \in \mathbf{C}\times\mathbf{S}^1$

(4.2) Definition. Let $Q = (q,z) \in \mathbf{C}\times\mathbf{S}^1$ and $\theta_1 \in \mathbf{R}$ satisfy $e^{i\theta_1} = z$. We denote by $\mathcal{U}_c$, $\mathcal{U}_d$ and $\mathcal{T}$ the subspaces of $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ consisting of all condensed, diffuse and critical curves, respectively.

(4.3) Theorem B. *The subspace $\mathcal{U}_c \subset \mathcal{L}^{+1}_{-1}(Q;\theta_1)$ consisting of all condensed curves is either empty or homeomorphic to $\mathbf{E}$, hence contractible.*

Recall that $\mathbf{E}$ denotes the separable Hilbert space. In what follows we shall call a function ϕ of a real variable *increasing* (resp. *decreasing*) if $x < y$ (resp. $x > y$) implies that $\phi(x) \le \phi(y)$. The previous theorem will be derived as a corollary of the following result.

(4.4) Proposition. *Let $\kappa_0 > 0$ and $\hat{\mathcal{U}}_c \subset \hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1)$ be the subspace consisting of all condensed curves. If $\hat{\mathcal{U}}_c \neq \emptyset$, then there exists a continuous $H\colon [0,1]\times\hat{\mathcal{U}}_c \to \hat{\mathcal{U}}_c$ such that, for all $\gamma \in \hat{\mathcal{U}}_c$:*

(i) *$H(1,\gamma) = \gamma$ and $H(0,\gamma) = \gamma_0$ is independent of γ.*
(ii) *The amplitude of $\gamma_s = H(s,\gamma)$ is an increasing function of $s \in [0,1]$.*
(iii) *The length of $\gamma_s = H(s,\gamma)$ is an increasing function of $s \in [0,1]$.*

In particular, $\hat{\mathcal{U}}_c$ is contractible. Moreover, γ_0 is the unique curve of minimal length in $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q)$.

Although this proposition plays only an auxiliary role here, we believe that it may be important for other purposes, e.g., for calculating the minimal length of curves in $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q)$.[†] We shall describe the effect of H on a single curve $\gamma \in \hat{\mathcal{U}}_c$ and then derive its main properties separately as lemmas. First we record two basic results which will be used to show that $H(0,\gamma)$ is independent of γ.

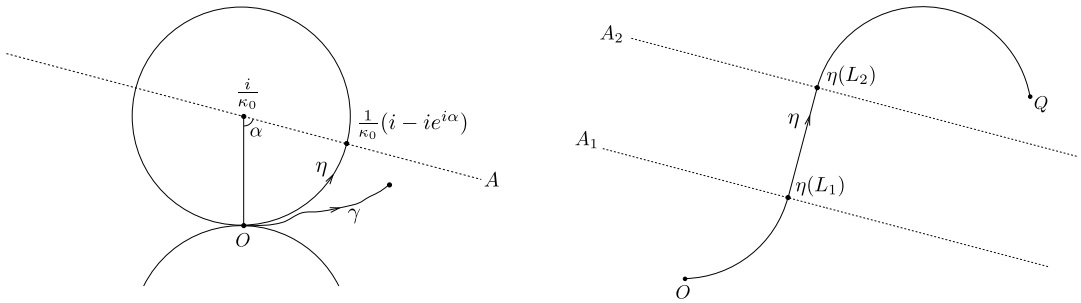


FIGURE 4. An illustration of (4.5) and (4.6).

(4.5) Lemma. *Let $Q = (q,z) \in \mathbf{C}\times\mathbf{S}^1$, $\gamma \in \hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q)$ and L be the length of γ. Suppose that q lies on the line through $\frac{i}{\kappa_0}$ having direction $-ie^{i\alpha}$, for some $\alpha \in [0,\pi)$. Then $L \ge \frac{\alpha}{\kappa_0}$ and equality holds if and only if γ is a reparametrization of the arc of the circle centered at $\frac{i}{\kappa_0}$ joining 0 to $\frac{1}{\kappa_0}(i - ie^{i\alpha})$.*

Proof. We lose no generality in assuming that $\kappa_0 = 1$. If $\alpha = 0$, there is nothing to prove, so suppose $\alpha \in (0,\pi)$. Let $\gamma\colon [0,L] \to \mathbf{C}$ be parametrized by arc-length, and let $\eta\colon [0,\alpha] \to \mathbf{C}$ be given by

$$\eta(s) = \int_0^s e^{i\sigma}\, d\sigma = i - ie^{is} \quad (s \in [0,\alpha]),$$

[†]The subspace $\hat{\mathcal{U}}_c$ will not be used outside of this subsection.

so that η is the parametrization by arc-length of the arc of circle described in (4.5), see fig. 4(a). Set

$$f\colon [0,L]\to\mathbf{R},\quad f(s)=\langle\gamma(s)-i,e^{i\alpha}\rangle,$$
$$g\colon [0,\alpha]\to\mathbf{R},\quad g(s)=\langle\eta(s)-i,e^{i\alpha}\rangle.$$

Let A denote the line in the statement. Note that $f(s)=0$ if and only if $\gamma(s)\in A$. We need to prove that $f(s)<0$ for all $s\in[0,\alpha)\cap[0,L]$. Let θ_γ be the argument of $\mathbf{t}_\gamma$ satisfying $\theta_\gamma(0)=0$. Then

$$f'(s)=\langle e^{i\theta_\gamma(s)},e^{i\alpha}\rangle=\cos(\alpha-\theta_\gamma(s))\ \text{ and }\ g'(s)=\langle e^{is},e^{i\alpha}\rangle=\cos(\alpha-s), \tag{18}$$

We have $f(0)=g(0)$. Since $g(s)<0$ for all $s\in[0,\alpha)$, it suffices to establish that $f'(s)\le g'(s)$ for all $s\in[0,\alpha]\cap[0,L]$. By the definition of $\hat{\mathcal{L}}^{+1}_{-1}(Q)$, θ_γ is 1-Lipschitz. Hence, $|\theta_\gamma(s)|\le s$ for all $s\in[0,L]$. Consequently,

$$\alpha-s\le\alpha-\theta_\gamma(s)\le\alpha+s\ \text{ for all } s\in[0,L].$$

In particular, $\alpha-\theta_\gamma(s)\in[0,2\pi]$ for all $s\in[0,\alpha]\cap[0,L]$. Since the cosine is decreasing over $[0,\pi]$, it follows immediately from (18) that if $\alpha-\theta_\gamma(s)\le\pi$, then $f'(s)\le g'(s)$. On the other hand, if $\alpha-\theta_\gamma(s)\in[\pi,2\pi]$, then from $\alpha-\theta_\gamma(s)\le\alpha+s$, we obtain that

$$\cos(\alpha-\theta_\gamma(s))\le\cos(\alpha+s)\le\cos(\alpha-s),$$

the latter inequality coming from $\alpha\in(0,\pi)$ and $s\in[0,\alpha]$. Thus, $f'(s)\le g'(s)$ in this case also. We conclude that $f(s)\le g(s)<0$ for all $s\in[0,\alpha)\cap[0,L]$. In particular, $L\ge\alpha$, as $\gamma(L)\in A$.

If $f(\alpha)=g(\alpha)=0$, then we must have $f'=g'$, that is, $\theta_\gamma(s)=s$ for all $s\in[0,\alpha]$. Thus, in this case, $\gamma|_{[0,\alpha]}$ is a reparametrization of $\eta|_{[0,\alpha]}$. □

(4.6) Corollary. *Suppose that $\eta\in\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q)$ is a concatenation of an arc of circle of curvature $\pm\kappa_0$, a line segment, and another arc of circle of curvature $\pm\kappa_0$, where some of these may be degenerate and both arcs have length less than $\frac{\pi}{\kappa_0}$. Then η is the unique curve in $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q)$ of minimal length.*[†]

Proof. Let $\eta\colon[0,L]\to\mathbf{C}$ be parametrized by arc-length, with $\eta|_{[0,L_1]}$, $\eta|_{[L_1,L_2]}$ and $\eta|_{[L_2,L]}$ corresponding to the first arc, line segment and second arc, respectively (see fig. 4(b)). Let A_i be the line perpendicular to $\eta'(L_i)$ passing through $\eta(L_i)$, $i=1,2$. Notice that A_1 and A_2 are parallel (or equal). Suppose that $\gamma\colon[0,M]\to\mathbf{C}$ is another curve in $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q)$, parametrized by arc-length. Let

$$M_1=\inf\big\{s\in[0,M]:\gamma(s)\in A_1\big\},\quad M_2=\sup\big\{s\in[0,M]:\gamma(s)\in A_2\big\}.$$

By (4.5), we have $M_1\ge L_1$ and $M-M_2\ge L-L_2$. It is clear that $M_2-M_1\ge L_2-L_1$ since any path joining a point of A_1 to a point of A_2 must have length greater than or equal to the distance between these lines. Hence, $M\ge L$. Furthermore, if equality holds, then $M_1=L_1$, $M-M_2=L-L_2$ and $M_2-M_1=L_2-L_1$. By (4.5), the two former equalities imply that $\gamma|_{[0,M_1]}=\eta|_{[0,L_1]}$ and $\gamma|_{[M_2,M]}=\eta|_{[L_2,L]}$. The condition $M_2-M_1=L_2-L_1$ then implies that $\gamma|_{[M_1,M_2]}$ must coincide with the line segment $\eta|_{[L_1,L_2]}$. □

(4.7) Construction. Let $\gamma\in\hat{\mathcal{U}}_c$, $\theta\colon[0,1]\to\mathbf{R}$ be the argument of $\mathbf{t}_\gamma$ satisfying $\theta(0)=0$. A number $\varphi\in(-\frac{\pi}{2},\frac{\pi}{2})$ will be called an *axis* of γ if $\langle\mathbf{t}_\gamma(t),e^{i\varphi}\rangle>0$ for all $t\in[0,1]$. Since γ is condensed, the set of all axes of γ is an open interval. The most natural axis, and the center of this interval, is

$$\bar\theta=\frac{1}{2}\Big(\max_{t\in[0,1]}\theta(t)+\min_{t\in[0,1]}\theta(t)\Big). \tag{19}$$

Let φ be any axis of γ. Rotating around the origin through φ and writing $\gamma(t)=(x(t),y(t))$, the hypothesis that $\langle\mathbf{t}_\gamma,e^{i\varphi}\rangle>0$ becomes equivalent to the fact that $\dot x$ is bounded and positive over $[0,1]$. Let

$$\gamma(x)=(x,y(x))\quad(x\in[0,b])$$

be the reparametrization of γ by x and define

$$f\colon[0,b]\to\mathbf{R}\quad\text{by}\quad f(x)=\dot y(x).$$

[†]Compare this result to prop. 9 in [2]. Their proofs are essentially the same.

Let $f_s\colon [0,b] \to \mathbf{R}$ ($s \in [0,1]$) be a family of absolutely continuous functions and set

$$\gamma_s(x) = \Big(x, \int_0^x f_s(u)\,du\Big) \quad (x \in [0,b]).$$

A straightforward computation shows that the curvature of γ_s is given by

$$\kappa_{\gamma_s}(x) = \frac{\dot f_s(x)}{[1+f_s(x)^2]^{\frac{3}{2}}} \quad (x \in [0,b]).$$

Therefore, γ_s lies in $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1)$ if and only if f_s satisfies:

(i) $|\dot f_s(x)| \le \kappa_0[1+f_s(x)^2]^{\frac{3}{2}}$ for almost every $x \in [0,b]$;
(ii) $f_s(0) = r_0 := \dot y(0)$ and $f_s(b) = r_b := \dot y(b)$ (that is, $\mathbf{t}_{\gamma_s}(0) = \mathbf{t}_\gamma(0)$ and $\mathbf{t}_{\gamma_s}(b) = \mathbf{t}_\gamma(b)$);
(iii) $\int_0^b f_s(x)\,dx = A_0 := y(b) - y(0)$ (that is, $\gamma_s(b) = \gamma(b)$).

We will now produce a homotopy of $f = f_1$ through absolutely continuous functions satisfying (i)–(iii). Define

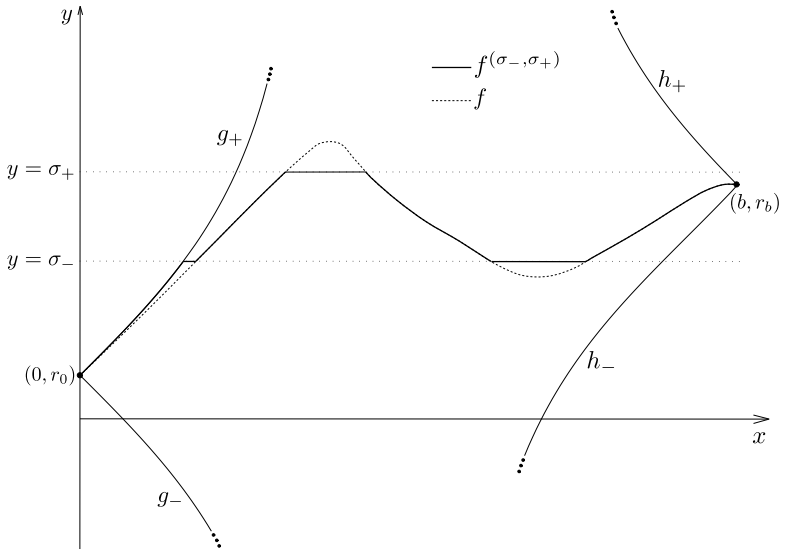


FIGURE 5. An illustration of (4.7).

$$\alpha_\pm = \mp\frac{r_0}{\sqrt{1+r_0^2}}, \quad g_\pm(x) = \pm\frac{\kappa_0 x - \alpha_\pm}{\sqrt{1-(\kappa_0 x - \alpha_\pm)^2}} \quad \text{for} \quad x \in \Big(\frac{\alpha_\pm - 1}{\kappa_0}, \frac{\alpha_\pm + 1}{\kappa_0}\Big)$$

(see fig. 5) and, similarly,

$$\beta_\pm = \kappa_0 b \pm \frac{r_b}{\sqrt{1+r_b^2}}, \quad h_\pm(x) = \mp\frac{\kappa_0 x - \beta_\pm}{\sqrt{1-(\kappa_0 x - \beta_\pm)^2}} \quad \text{for} \quad x \in \Big(\frac{\beta_\pm - 1}{\kappa_0}, \frac{\beta_\pm + 1}{\kappa_0}\Big).$$

The functions $g_\pm$ are the solutions of the differential equations $\dot g = \pm\kappa_0(1+g^2)^{\frac{3}{2}}$ with $g(0) = r_0$. Similarly, $h_\pm$ are the solutions of the differential equations $\dot h = \mp\kappa_0(1+h^2)^{\frac{3}{2}}$ with $h(b) = r_b$. Extend their domains to all of $[0,b]$ by setting

$$g_\pm(x) = \pm\infty \quad \text{if} \quad x \ge \frac{\alpha_\pm + 1}{\kappa_0} \quad \text{and} \quad h_\pm(x) = \pm\infty \quad \text{if} \quad x \le \frac{\beta_\pm - 1}{\kappa_0}.$$

Since the curvature of $\gamma = \gamma_1$ takes values in $[-\kappa_0, +\kappa_0]$, condition (i) applied to $f = f_1$ gives:

$$g_-(x),\, h_-(x) \le f(x) \le g_+(x),\, h_+(x) \quad \text{for all } x \in [0,b]. \tag{20}$$

Let

$$m_- = \min_{x\in[0,b]} f(x), \quad m_+ = \max_{x\in[0,b]} f(x) \quad \text{and} \quad \Delta = \big\{(\sigma_-,\sigma_+) \in [m_-,m_+] : \sigma_- \le \sigma_+\big\}.$$

For $(\sigma_-, \sigma_+) \in \Delta$, let $f^{(\sigma_-,\sigma_+)}\colon [0,b] \to \mathbf{R}$ be given by

$$f^{(\sigma_-,\sigma_+)}(x) = \text{median}\big(h_-(x)\,,\, g_-(x)\,,\, \sigma_-\,,\, f(x)\,,\, \sigma_+\,,\, g_+(x)\,,\, h_+(x)\big) \tag{21}$$

(cf. fig. 5). The functions $f^{(\sigma_-,\sigma_+)}$ automatically satisfy conditions (i) and (ii). Define $A\colon \Delta \to \mathbf{R}$ to be the area under the graph of $f^{(\sigma_-,\sigma_+)}$:

$$A(\sigma_-, \sigma_+) = \int_0^b f^{(\sigma_-,\sigma_+)}(x)\,dx.$$

It is immediate from (21) that:

(I) A is increasing as a function of σ_- (resp. σ_+);
(II) A is a Lipschitz function of (σ_-, σ_+). In fact,

$$|A(\sigma_- + u, \sigma_+ + v) - A(\sigma_-, \sigma_+)| \le b(|u| + |v|).$$

By (I), for each $s \in [0,1]$, the set

$$\big\{(\sigma_-, \sigma_+) \in \Delta : A(\sigma_-, \sigma_+) = A_0 \text{ and } \sigma_+ - \sigma_- = (m_+ - m_-)s\big\}$$

is an interval of the latter line in the (σ_-, σ_+)-plane. Let $(\sigma_-(s), \sigma_+(s))$ be the coordinates of the center of this interval. By (II), $\sigma_-(s)$ and $\sigma_+(s)$ are continuous (even Lipschitz), and (I) implies that σ_- is a decreasing, while σ_+ is an increasing function of $s \in [0,1]$. The functions

$$f_s\colon [0,b] \to \mathbf{R}, \quad f_s = f^{(\sigma_-(s),\sigma_+(s))}$$

satisfy all of conditions (i)–(iii) by construction. We repeat their definition for convenience:

$$\begin{aligned} f_s(x) &= \text{median}\big(h_-(x)\,,\, g_-(x)\,,\, \sigma_-(s)\,,\, f(x)\,,\, \sigma_+(s)\,,\, g_+(x)\,,\, h_+(x)\big), \\ \gamma_s(x) &= \Big(x, \int_0^x f_s(u)\,du\Big) \quad (x \in [0,b]). \end{aligned} \tag{22}$$

We will denote $\sigma_+(0) = \sigma_-(0)$ by σ_0. The monotonicity of σ_-, σ_+ implies that

$$\sigma_-(s) \le \sigma_0 \le \sigma_+(s) \quad \text{ for all } s \in [0,1]. \tag{23}$$

□

(4.8) *Remark.* We deduce from (20) and (22) that

$$f_0 = \text{median}(h_-, g_-, \sigma_0, g_+, h_+)$$

is independent of f. The graph of f_0 is composed of at most three parts: a piece of the graph of g_- or g_+, a piece of the graph of the constant function $y = \sigma_0$, and a piece of the graph of h_- or h_+. The corresponding curve γ_0 is thus the concatenation of an arc of circle of curvature $\pm\kappa_0$, a line segment and another arc of circle of curvature $\pm\kappa_0$, though some of these may degenerate to a point. It is an immediate consequence of (4.6) that γ_0 is independent of γ and of the chosen axis φ.

(4.9) Lemma. *Let φ be an axis of $\gamma \in \hat{\mathcal{U}}_c$ and $s \mapsto \gamma_s$ ($s \in [0,1]$) be the deformation described in (4.7). Then $\gamma_0 \in \hat{\mathcal{U}}_c$ is the unique curve of minimal length in $\hat{\mathcal{L}}_{-\kappa_0}^{+\kappa_0}(Q)$.* □

Remark. Notice that this proves Dubins' theorem I in [2] in the case where $\hat{\mathcal{L}}_{-\kappa_0}^{+\kappa_0}(Q)$ contains condensed curves. Furthermore, given Q and κ_0, we can use (4.7) to describe γ_0 explicitly.

(4.10) Lemma. *Let $A_+ = \big\{x \in [0,b] : f(x) \ge \sigma_0\big\}$ and $A_- = \big\{x \in [0,b] : f(x) \le \sigma_0\big\}$. Then $f_s(x)$ is an increasing (resp. decreasing) function of $s \in [0,1]$ if $x \in A_+$ (resp. A_-). Moreover, for all $s \in [0,1]$, $f_s(x) \ge \sigma_0$ if $x \in A_+$ and $f_s(x) \le \sigma_0$ if $x \in A_-$.*

Proof. Suppose that $x \in A_+$. From (20) and (23), we deduce that

$$g_-(x),\, h_-(x),\, \sigma_-(s) \le f(x) \le g_+(x),\, h_+(x).$$

Hence, $f_s(x) = \min\{\sigma_+(s), f(x)\} \ge \sigma_0$ and $f_s(x)$ increases with s since $\sigma_+(s)$ does. The proof for $x \in A_-$ is analogous. □

(4.11) Corollary. *Let $m_-(s) = \min_{x\in[0,b]} f_s(x)$ and $m_+(s) = \max_{x\in[0,b]} f_s(x)$. Then $m_+(s)$ is an increasing and $m_-(s)$ a decreasing function of $s \in [0,1]$.* □

(4.12) Lemma. *Let $\gamma \in \hat{\mathcal{U}}_c$ and $s \mapsto \gamma_s \in \hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1)$ be the deformation described in (4.7). Let ω_s denote the amplitude of γ_s. Then ω_s is an increasing function of s; in particular, γ_s is condensed (i.e., $\gamma_s \in \hat{\mathcal{U}}_c$) for all $s \in [0,1]$.*

Proof. Let φ be the axis of γ chosen for the construction. Recall that, by definition,

$$\omega_s = \max_{x\in[0,b]} \theta_s(x) - \min_{x\in[0,b]} \theta_s(x) \quad (s \in [0,1]), \tag{24}$$

where θ_s is the argument of $\mathbf{t}_{\gamma_s}$ such that $\theta_s(0) = 0$. By (22),

$$f_s(x) = \tan(\theta_s(x) - \varphi). \tag{25}$$

Because the tangent is an increasing function, (4.11) immediately implies that ω_s is increasing. □

Remark. Although γ_0 has minimal amplitude in $\hat{\mathcal{U}}_c$ by the previous lemma, there may be other curves in $\hat{\mathcal{U}}_c$ with the same amplitude. This is the case, for instance, for the curves γ_0 and γ corresponding to the functions f and f_0 of fig. 5.

(4.13) Lemma. *Let $\gamma \in \hat{\mathcal{U}}_c$ and $s \mapsto \gamma_s$ the deformation described in (4.7). Then the length of γ_s is an increasing function of $s \in [0,1]$.*

Proof. Let $\lambda\colon \mathbf{R} \to \mathbf{R}$ be given by $\lambda(u) = (1+u^2)^{\frac{1}{2}}$. A straightforward computation shows that

$$\lambda''(u) = (1+u^2)^{-\frac{3}{2}} > 0 \quad \text{for all } u \in \mathbf{R}. \tag{26}$$

Moreover, by definition (22), the length L_s of γ_s is given by

$$L_s = \int_0^b (\lambda \circ f_s)(x)\, dx.$$

Let $s_1 \le s_2 \in [0,1]$, A_+, A_- be as in (4.10) and

$$\begin{aligned} B_+ &= \big\{(x,y) \in [0,b]\times\mathbf{R} : f_{s_1}(x) \le y \le f_{s_2}(x)\big\} \quad \text{and} \\ B_- &= \big\{(x,y) \in [0,b]\times\mathbf{R} : f_{s_2}(x) \le y \le f_{s_1}(x)\big\}. \end{aligned}$$

Using (4.10), we deduce that

$$\begin{aligned} L_{s_2} - L_{s_1} &= \int_0^b (\lambda\circ f_{s_2})(x) - (\lambda \circ f_{s_1})(x)\, dx \\ &= \left(\int_{A_+} + \int_{A_-}\right)(\lambda\circ f_{s_2})(x) - (\lambda\circ f_{s_1})(x)\, dx \\ &= \left(\int_{B_+} - \int_{B_-}\right)\lambda'(y)\, dy\, dx \\ &\ge \left(\int_{B_+} - \int_{B_-}\right)\lambda'(\sigma_0)\, dy\, dx \qquad \text{by (26)} \\ &= \lambda'(\sigma_0)\big(\operatorname{area}(f_{s_2}) - \operatorname{area}(f_{s_1})\big) = 0 \qquad \text{by the definition of } f_s. \end{aligned}$$

Therefore, L_s is an increasing function of $s \in [0,1]$. □

We are finally ready to prove (4.4) and (4.3).

Proof of (4.4). For each $\gamma \in \hat{\mathcal{U}}_c$, let

$$\varphi_\gamma = \frac{1}{2}\Big(\max_{t\in[0,1]} \theta_\gamma(t) + \min_{t\in[0,1]} \theta_\gamma(t)\Big), \tag{27}$$

where $\theta_\gamma\colon [0,1] \to \mathbf{R}$ is the argument of $\mathbf{t}_\gamma$ satisfying $\theta_\gamma(0) = 0$. It is clear that φ_γ depends continuously on $\gamma \in \hat{\mathcal{U}}_c$. Define $H\colon [0,1]\times\hat{\mathcal{U}}_c \to \hat{\mathcal{U}}_c$ by $H(s,\gamma) = \gamma_s$, where γ_s is the curve (22) constructed in (4.7) with chosen axis φ_γ. Then part (ii) of (4.4) follows from (4.12), and part (iii) from (4.13). The last assertion of (4.4) and part (i) were established in (4.8). □

Proof of (4.3). Assume that $\mathcal{U}_c$ is nonempty. It is certainly open in $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$. Hence, by (1.12), it suffices to prove that $\mathcal{U}_c$ is weakly contractible. Let K be a compact manifold and $F\colon K \to \mathcal{U}_c$, $a \mapsto \gamma^a$, be a continuous map. Using (1.15), we may assume that the image of F is contained in (the image under set inclusion of) $\mathcal{C}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1)$ for some $\kappa_0 \in (0,1)$. By (2.4), we have continuous injections

$$\mathcal{C}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1) \to \hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1) \to \mathcal{L}^{+1}_{-1}(Q;\theta_1).$$

Let $G\colon [0,1]\times K \to \mathcal{L}^{+1}_{-1}(Q;\theta_1)$ map (s,a) to (the image under set inclusion of) $H(s,\gamma^a)$, with H as in (4.4). Then G is a null-homotopy of F in $\mathcal{U}_c$. □

The next two lemmas will only be needed in later sections.

(4.14) Lemma. *Suppose that there exists $\hat\omega \in (0,\pi)$ such that if $\gamma \in \hat{\mathcal{U}}_c \subset \hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1)$ then its amplitude ω_γ satisfies $\omega_\gamma \le \hat\omega$. Let $L(\eta)$ denote the length of η. Then $\sup_{\gamma\in\hat{\mathcal{U}}_c} L(\gamma)$ is finite. In particular, the images of $\gamma \in \hat{\mathcal{U}}_c$ are all contained in some bounded subset of $\mathbf{C}$.*

Proof. Let $\gamma \in \hat{\mathcal{U}}_c$ and φ_γ be as in (27). By hypothesis, the image of $\theta_\gamma\colon [0,1]\to\mathbf{R}$ is contained in $\left[\varphi_\gamma - \frac{\hat\omega}{2}\,,\, \varphi_\gamma + \frac{\hat\omega}{2}\right]$. Let $f\colon [0,b]\to\mathbf{R}$ be the function corresponding to γ and the axis φ_γ, in the notation of (4.7). Note that $b = \left\langle e^{i\varphi_\gamma}, q\right\rangle \le |q|$, where q is the $\mathbf{C}$-coordinate of Q. By (25),

$$|f(x)| \le \tan\left(\frac{\hat\omega}{2}\right) \text{ for all } x \in [0,b].$$

Therefore, the length $L(\gamma)$ of γ satisfies

$$L(\gamma) = \int_0^b \sqrt{1+f(x)^2}\,dx \le b\sec\left(\frac{\hat\omega}{2}\right) \le |q|\sec\left(\frac{\hat\omega}{2}\right).$$

Consequently, the image of γ is contained in the closed disk centered at 0 of radius $|q|\sec(\hat\omega/2)$. □

(4.15) Lemma. *Let $\hat{\mathcal{U}}_c \subset \hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q;\theta_1)$ and $H\colon [0,1]\times\hat{\mathcal{U}}_c \to \hat{\mathcal{U}}_c$ be the deformation described in (4.4) and (4.7). Suppose that $\theta_1 = 0$. Then $\omega_0 < \omega_1$ unless $\gamma_1 = \gamma_0$.*

Proof. It is obvious that $\omega_1 = \omega_0$ if $\gamma_1 = \gamma_0$. The condition $\theta_1 = 0$ is equivalent to $r_0 = r_b$, in the notation of (4.7). Suppose without loss of generality that $\sigma_0 \ge r_0$, so that $m_+(0) = \sigma_0$.

If $m_+(1) \le \sigma_0$, then $A_- = [0,b]$. Hence, by (4.10), $f_1(x) \le f_0(x)$ for all $x \in [0,b]$. Since f_1 and f_0 have the same area, we conclude that $f_1 = f_0$, that is, $\gamma_1 = \gamma_0$.

By (4.11), $m_-(1) \le m_-(0)$. Hence, if $m_+(1) > \sigma_0 = m_+(0)$, then $\omega_0 < \omega_1$ by (24) and (25). □

Existence of condensed curves. The question of whether $\mathcal{U}_c \ne \emptyset$ is settled by means of an elementary geometric construction. In all that follows, $C_r(a)$ denotes the circle of radius $r > 0$ centered at $a \in \mathbf{C}$.

(4.16) Proposition. *Let $\theta_1 \in [0,\pi)$ be fixed, $z = e^{i\theta_1}$ and $Q = (q,z) \in \mathbf{C}\times\mathbf{S}^1$. Let $R_{\mathcal{U}_c}$ be the open region of the plane which does* not *contain the negative real axis and which is bounded by the shortest arcs of the circles $C_2(\pm(i+iz))$ joining $i - iz$ to $i - iz \pm 2(i+iz)$ and their tangent lines at the latter points. Then $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ contains condensed curves if and only if $q \in R_{\mathcal{U}_c}$.*

It is clear from the definition of condensed that $\mathcal{U}_c \subset \mathcal{L}^{+1}_{-1}(Q;\theta_1)$ is empty if $|\theta_1| \ge \pi$. In other words, all condensed curves in $\mathcal{L}^{+1}_{-1}(Q)$ must be contained in the subspace $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ with θ_1 the unique number in $(-\pi,\pi)$ satisfying $e^{i\theta_1} = z$ (for $z \ne -1$). We have assumed that $\theta_1 \in [0,\pi)$ just to simplify the statement. If $\theta_1 \in (-\pi,0]$, the only difference is that the points bounding the arcs of $C_2(\pm(i+iz))$ are now $-i+iz$ and $-i+iz\pm 2(i+iz)$. The proof is analogous to the one given below. Alternatively, it can be deduced from the proposition by applying a reflection across the x-axis. When $\theta_1 = 0$, the statement is ambiguous; the arcs of circles which bound $R_{\mathcal{U}_c}$ are centered at $\pm 2i$, bounded by 0 and $\pm 4i$, and pass through the points $2\pm 2i$, respectively.

Proof of (4.16). Let $\eta\colon [0,1]\to\mathbf{C}$ be condensed and let $\theta_\eta\colon [0,1]\to\mathbf{R}$ be the argument of $\mathbf{t}_\eta$ satisfying $\theta_\eta(0) = 0$. Observe that

$$\left(\min_{t\in[0,1]} \theta_\eta(t)\right) \in [\theta_1 - \pi, 0] \quad\text{and}\quad \left(\max_{t\in[0,1]} \theta_\eta(t)\right) \in [\theta_1, \pi].$$

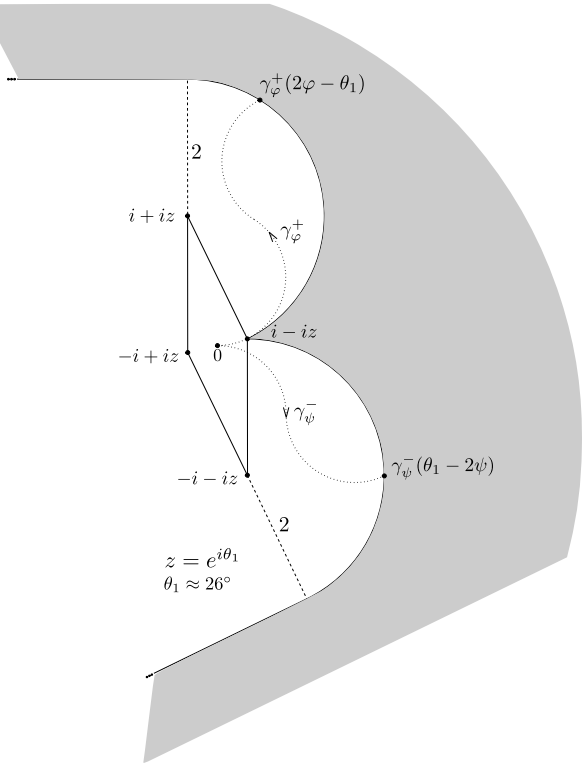


FIGURE 6. Let $\theta_1 \in [0,\pi)$ be fixed and $Q=(q,z)$, where $z=e^{i\theta_1}$. There exist condensed curves in $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ if and only if q belongs to the open gray region.

The proof relies on the study of the following curves. For each $\varphi \in [\theta_1,\pi]$, define $\gamma_\varphi^+\colon [0,2\varphi-\theta_1] \to \mathbf{C}$ to be the unique curve parametrized by arc-length satisfying:

$$\gamma_\varphi^+(0)=0 \quad \text{and} \quad \mathbf{t}_{\gamma_\varphi^+}(s)=\begin{cases} e^{is} & \text{if } s\in[0,\varphi] \\ e^{i(2\varphi-s)} & \text{if } s\in[\varphi,2\varphi-\theta_1] \end{cases}$$

Then γ_φ^+ is the concatenation of two arcs of circles of radius 1,

$$\min_{t\in[0,1]} \theta_{\gamma_\varphi^+}(t)=0, \quad \max_{t\in[0,1]} \theta_{\gamma_\varphi^+}(t)=\varphi, \quad \mathbf{t}_{\gamma_\varphi^+}(2\varphi-\theta_1)=z \quad \text{and}$$

$$\gamma_\varphi^+(2\varphi-\theta_1)=\int_0^{\varphi} e^{is}\,ds+\int_{\varphi}^{2\varphi-\theta_1} e^{i(2\varphi-s)}\,ds = i+iz-2ie^{i\varphi}.$$

Thus, as φ increases from θ_1 to π, the endpoints of the γ_φ^+ trace out the arc of $C_2(i+iz)$ bounded by $i-iz$ and $3i+iz$. Further, the tangent line to $C_2(i+iz)$ at $\gamma_\varphi^+(2\varphi-\theta_1)$ is parallel to $e^{i\varphi}$, for it must be orthogonal to $-2ie^{i\varphi}$.

Similarly, for each $\psi \in [\theta_1-\pi,0]$, let $\gamma_\psi^-\colon [0,\theta_1-2\psi] \to \mathbf{C}$ be the curve, parametrized by arc-length, which satisfies

$$\gamma_\psi^-(0)=0 \quad \text{and} \quad \mathbf{t}_{\gamma_\psi^-}(s)=\begin{cases} e^{-is} & \text{for } s\in[0,-\psi] \\ e^{i(2\psi+s)} & \text{for } s\in[-\psi,\theta_1-2\psi] \end{cases}$$

Then γ_ψ^- is the concatenation of two arcs of circles of radius 1, $\mathbf{t}_{\gamma_\psi^-}(\theta_1-2\psi)=z$ for all $\psi\in[\theta_1-\pi,0]$ and as ψ decreases from 0 to $\theta_1-\pi$, the endpoints of the γ_ψ^- traverse the arc of $C_2(-i-iz)$ bounded by $i-iz$ and $-i-3iz$. Moreover, the tangent line to this circle at $\gamma_\psi^-(\theta_1-2\psi)$ is parallel to $e^{i\psi}$.

Any $q\in\overline{R}_{\mathcal{U}_c}$ is the endpoint of a curve of one of the following three types:

(i) The concatenation of a γ_φ^+ or a γ_ψ^- with a line segment of direction z.

(ii) The concatenation of $\gamma_\pi^+|_{[0,\pi]}$, a line segment of length $\ell \geq 0$ having direction -1, the arc $-\ell + \gamma_\pi^+|_{[\pi, 2\pi-\theta_1]}$, and a line segment of direction z.

(iii) The concatenation of $\gamma_{\theta_1-\pi}^-|_{[0,\pi-\theta_1]}$, a line segment of length $\ell_1 \geq 0$ of direction $-z$, the arc $-\ell_1 z + \gamma_{\theta_1-\pi}^-|_{[\pi-\theta_1, \frac{3\pi}{2}-\theta_1]}$, a line segment of length $\ell_2 \geq 0$ and direction $-iz$, the arc $-\ell_1 z - \ell_2 iz + \gamma_{\theta_1-\pi}^-|_{[\frac{3\pi}{2}-\theta_1, 2\pi-\theta_1]}$, and a line segment of direction z.

The curves which we have described have curvature equal to ± 1 over intervals of positive measure and may be critical. Nevertheless, for any $q \in R_{\mathcal{U}_c}$, we can find a condensed $\gamma \in \mathcal{L}_{-1}^{+1}(Q;\theta_1)$ by composing one of these curves with a dilatation through a factor $c > 1$, with c close to 1 if q lies close to $\partial R_{\mathcal{U}_c}$, and by avoiding the argument π (for a curve of type (i)) or $\theta_1 - \pi$ (for a curve of type (iii)).

Conversely, suppose that $\mathcal{L}_{-1}^{+1}(Q)$ contains condensed curves. Let $\eta\colon [0,L] \to \mathbf{C}$ be such a curve, parametrized by arc-length, and let $\varphi = \max\theta$, where $\theta\colon [0,L] \to \mathbf{R}$ is an argument of $\mathbf{t}_\eta$ satisfying $\theta(0) = 0$. Define

$$g\colon [0,L] \to \mathbf{R} \quad \text{by} \quad g(s) = \left\langle \eta(s) - \gamma_\varphi^+(2\varphi - \theta_1)\,,\, ie^{i\varphi} \right\rangle.$$

Note that $g(s) > 0$ if and only if $\eta(s)$ lies to the left of the line through $\gamma_\varphi^+(2\varphi-\theta_1) \in C_2(i+iz)$ having direction $e^{i\varphi}$; we have already seen that this line is tangent to this circle at this point. We claim that $g(L) < 0$. Since η is admissible, θ is an absolutely continuous function by (2.2), and $|\theta'| = |\kappa_\eta| < 1$ almost everywhere by eq. (12). Moreover, $\theta(s) \in [\varphi - \pi, \varphi]$ for all s because η is condensed. Hence,

$$g'(s) = \left\langle e^{i\theta(s)}, ie^{i\varphi} \right\rangle = \cos\left(\theta(s) - \varphi - \tfrac{\pi}{2}\right) \leq 0 \quad \text{for all } s \in [0,L]. \tag{28}$$

Let $J_i = (a_i, b_i) \subset (0,L)$ $(i = 1,2,3)$ be disjoint intervals such that:

(I) $\theta(a_1) = 0$ and $\theta(b_1) = \theta_1$;
(II) $\theta(a_2) = \theta_1$ and $\theta(b_2) = \varphi$;
(III) $\theta(a_3) = \varphi$ and $\theta(b_3) = \theta_1$.

Such intervals exist because θ is a continuous function satisfying $\theta(0) = 0$, $\theta_1 \leq \varphi = \max\theta$ and $\theta(L) = \theta_1$. Let λ denote the Lebesgue measure on $\mathbf{R}$. Fix i and let $[\alpha,\beta]$ be any nondegenerate subinterval of $\theta((a_i,b_i))$. Since θ is strictly 1-Lipschitz, if $S = \big\{s \in (a_i,b_i) : \alpha \leq \theta(s) \leq \beta\big\}$, then $\lambda(S) > \beta - \alpha$. Combining this with (28), we deduce that

$$\begin{aligned} g(L) - g(0) &\leq \left(\int_{a_1}^{b_1} + \int_{a_2}^{b_2} + \int_{a_3}^{b_3} \right) g'(s)\,ds \\ &< \int_0^{\theta_1} \left\langle e^{it}, e^{i\varphi} \right\rangle\, dt + 2\int_{\theta_1}^{\varphi} \left\langle e^{it}, e^{i\varphi} \right\rangle dt = \left\langle \gamma_\varphi^+(2\varphi-\theta_1), ie^{i\varphi} \right\rangle. \end{aligned}$$

Therefore, $g(L) < 0$ as claimed. Similarly, if $\psi = \min\theta$, then $\eta(L)$ lies on the side of the tangent to $C_2(-i-iz)$ at $\gamma_\psi^-(\theta_1 - 2\psi)$ which does not contain points of $C_2(-i-iz)$. It follows that $q = \eta(L) \in R_{\mathcal{U}_c}$. □

5. Diffuse curves

Throughout this section, let K denote a compact manifold, possibly with boundary. Also, let $Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$ be fixed (but otherwise arbitrary) and let $\mathcal{U}_d \subset \mathcal{L}_{-1}^{+1}(Q;\theta_1)$ denote the subset consisting of all diffuse curves in $\mathcal{L}_{-1}^{+1}(Q)$ having total turning θ_1, for some fixed $\theta_1 \in \mathbf{R}$ satisfying $e^{i\theta_1} = z$. Finally, let $O = (0,1) \in \mathbf{C} \times \mathbf{S}^1$ (O is the identity element of the group $UT\mathbf{C}$).

Our next objective is to prove that $\mathcal{U}_d$ is contractible. The idea behind the proof is very simple. If γ is diffuse, then we can "graft" a straight line segment of length greater than 4 onto γ, as illustrated in fig. 9. By the theorem of Dubins stated in the introduction, this segment can be deformed so that in the end an eight curve of large radius traversed a number n of times has been attached to it, as in fig. 8(e). These eights are then spread along the curve, as in fig. 8(f). If $n \in \mathbf{N}$ is large enough,

the spreading can be carried out within $\mathcal{L}^{+1}_{-1}(Q)$. The result is a curve which is so loose that the constraints on the curvature may be safely forgotten, allowing us to use the following fact.†

(5.1) Theorem (Smale). *Let $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$. Then $\mathcal{C}^{+\infty}_{-\infty}(Q)$ and $\mathcal{L}^{+\infty}_{-\infty}(Q)$ have $\aleph_0$ connected components, one for each $\theta_1 \in \mathbf{R}$ satisfying $e^{i\theta_1} = z$, all of which are contractible.*

Proof. For the space $\mathcal{C}^{+\infty}_{-\infty}(Q)$, the proof was discussed in the introduction. We may replace $\mathcal{C}^{+\infty}_{-\infty}(Q)$ by $\mathcal{L}^{+\infty}_{-\infty}(Q)$ using (1.16). □

(5.2) Lemma. *Let $P \in UT\mathbf{C}$. Then $\mathcal{C}^{+1}_{-1}(P, P)$ and $\mathcal{L}^{+1}_{-1}(P, P)$ have $\aleph_0$ connected components, one for each turning number $n \in \mathbf{Z}$, all of which are contractible.*

Proof. By (1.16), it suffices to prove the result for $\mathcal{C}^{+1}_{-1}(P, P)$. Let $\mathcal{C}_n \subset \mathcal{C}^{+1}_{-1}(P, P)$ denote the subset of all curves which have turning number n. Then each $\mathcal{C}_n$ is closed and open. Hence, to establish that $\mathcal{C}_n$ is a contractible component, it suffices, by (1.12 (b)), to prove that it is weakly contractible.

Recall that $\mathcal{C}^{+1}_{-1}(P, P) \approx \mathcal{C}^{+1}_{-1}(O)$, the homeomorphism coming from composing all curves with a suitable Euclidean motion. We may thus assume that $P = O$. Let K be a compact manifold and $f\colon K \to \mathcal{C}_n$ a continuous map. By (5.1), there exists a continuous $F\colon [0,1] \times K \to \mathcal{C}^{+\infty}_{-\infty}(O)$ such that $F_0 = f$ and F_1 is a constant map. Let

$$M = 2 \sup \big\{ |\kappa_{F(s,a)}(t)| : s, t \in [0,1],\ a \in K \big\}.$$

Given a curve γ, let $M\gamma$ denote the dilated curve $t \mapsto M\gamma(t)$. It is easy to see that $\kappa_{M\gamma} = \frac{\kappa_\gamma}{M}$. Hence, MF is a homotopy between Mf and a constant map within $\mathcal{C}^{+1}_{-1}(O)$. But f and Mf are homotopic within $\mathcal{C}^{+1}_{-1}(O)$ through $u \mapsto uf$ $(u \in [1, M])$. Therefore, f is null-homotopic. □

Loops and eights. We shall now explain how to attach loops and eights to a curve, and how to spread eights along it (fig. 8).

(5.3) Definition. We denote by $\alpha\colon \mathbf{R} \to \mathbf{C}$ the *loop* of radius 2 and by $\beta\colon \mathbf{R} \to \mathbf{C}$ the *eight* curve of the same radius (see figs. 8 (b) and (d)) given by:

$$\alpha(t) = 2i\big(1 - \exp(2\pi i t)\big)$$

$$\beta(t) = \begin{cases} \alpha(2t) & \text{for } t \in [\frac{m}{2}, \frac{m+1}{2}],\ m \equiv 0 \pmod 2 \\ -\alpha(-2t) & \text{for } t \in [\frac{m}{2}, \frac{m+1}{2}],\ m \equiv 1 \pmod 2 \end{cases} \quad (m \in \mathbf{Z}).$$

We shall also denote by $\alpha_n, \beta_n\colon [0,1] \to \mathbf{C}$ a loop (resp. eight) traversed $n \geq 1$ times: $\alpha_n(t) = \alpha(nt)$ and $\beta_n(t) = \beta(nt)$ $(t \in [0,1])$.

Note that $\alpha_n, \beta_n \in \mathcal{L}^{+1}_{-1}(O)$. The curvature of α_n is everywhere equal to $\frac{1}{2}$, and that of β_n equals $\pm\frac{1}{2}$ except at the $2n-1$ points where it is undefined. The turning number of α_n is n, and that of β_n is 0.

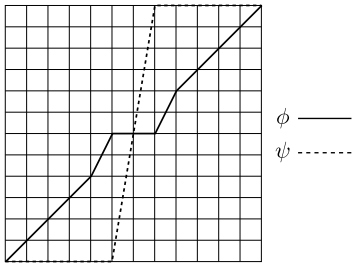


FIGURE 7.

†In [10] we also consider notions of condensed and diffuse for curves on the sphere $\mathbf{S}^2$. Although there is not much similarity between the respective definitions, the underlying ideas are analogous. In both cases, a diffuse curve is one which is homotopic to a "phone wire" curve, as in fig. 8 (f). The definitions of grafting and spreading loops/eights along a curve are essentially the same for both plane and spherical curves.

(5.4) Definition. Let $t_0 \in (0,1)$, $1 \le n \in \mathbf{N}$, $0 < 2\varepsilon < \min\{1-t_0, t_0\}$ and γ be an admissible plane curve. Define piecewise linear functions $\phi, \psi\colon [0,1] \to [0,1]$ (whose graphs are depicted in fig. 7) by:

$$\phi(t) = \begin{cases} t & \text{if} \quad t \notin [t_0 - 2\varepsilon, t_0 + 2\varepsilon] \\ 2t - t_0 + 2\varepsilon & \text{if} \quad t \in [t_0 - 2\varepsilon, t_0 - \varepsilon] \\ t_0 & \text{if} \quad t \in [t_0 - \varepsilon, t_0 + \varepsilon] \\ 2t - t_0 - 2\varepsilon & \text{if} \quad t \in [t_0 + \varepsilon, t_0 + 2\varepsilon] \end{cases} \qquad \psi(t) = \begin{cases} 0 & \text{if} \quad t \in [0, t_0 - \varepsilon] \\ \frac{t - t_0 + \varepsilon}{2\varepsilon} & \text{if} \quad t \in [t_0 - \varepsilon, t_0 + \varepsilon] \\ 1 & \text{if} \quad t \in [t_0 + \varepsilon, 1] \end{cases} \tag{29}$$

Define curves A_{γ,n,t_0}, B_{γ,n,t_0} (attaching loops, eights) and $S_{\gamma,n}\colon [0,1] \to \mathbf{C}$ (spreading eights) by:

$$\begin{aligned} A_{\gamma,n,t_0}(t) &= \Phi_\gamma(\phi(t))\alpha_n(\psi(t)) \\ B_{\gamma,n,t_0}(t) &= \Phi_\gamma(\phi(t))\beta_n(\psi(t)) \qquad (t \in [0,1]). \\ S_{\gamma,n}(t) &= \Phi_\gamma(t)\beta_n(t). \end{aligned}$$

Here $\Phi_\gamma\colon [0,1] \to \mathbf{C} \times \mathbf{S}^1$ is the frame of γ (as in (9)), but viewed as a curve in the *group* $UT\mathbf{C}$: Each $\Phi_\gamma(t)$ is an Euclidean motion, with $\Phi_\gamma(t)a = \gamma(t) + \mathbf{t}_\gamma(t)a$ for $a \in \mathbf{C}$. Different values of ε and t_0 yield curves which are homotopic in whichever space one is working with.

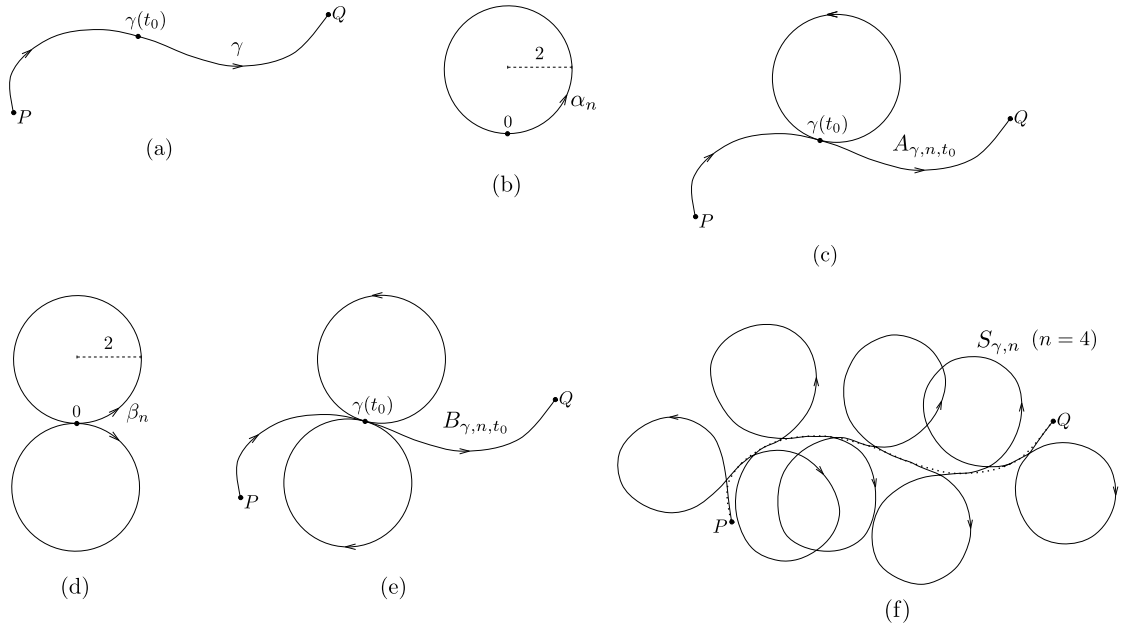


FIGURE 8.

(5.5) Lemma. *Let $t_0 \in (0,1)$, $1 \le n \in \mathbf{N}$ and γ be an admissible plane curve. Then:*

(a) *A_{γ,n,t_0}, B_{γ,n,t_0} and $S_{\gamma,n}$ have the same initial and final frames as γ.*
(b) *B_{γ,n,t_0} and $S_{\gamma,n}$ lie in the same connected component of $\mathcal{L}^{+\infty}_{-\infty}(P,Q)$ ($P = \Phi_\gamma(0)$, $Q = \Phi_\gamma(1)$).*
(c) *If $\gamma \in \mathcal{L}^{+1}_{-1}(P,Q)$, then A_{γ,n,t_0}, $B_{\gamma,n,t_0} \in \mathcal{L}^{+1}_{-1}(P,Q)$ also.*
(d) *Let $O = (0,1) \in \mathbf{C} \times \mathbf{S}^1$. Then α_1 and B_{α_1,n,t_0} lie in the same connected component of $\mathcal{L}^{+1}_{-1}(O)$ for all $n \ge 1$.*
(e) *If $f, g\colon K \to \mathcal{U}_d$ are continuous and homotopic within $\mathcal{U}_d$, then so are B_{f,n,t_0} and B_{g,n,t_0}.*
(f) *If γ is a reparametrization of α_1, then A_{γ,n,t_0} is a reparametrization of α_{n+1}.*

Proof. It is clear that A_{γ,n,t_0}, B_{γ,n,t_0} have the same initial and final frames as γ, since they agree with γ in neighborhoods of the endpoints of $[0,1]$. From the definition of $S_{\gamma,n}$, we find that

$$\dot{S}_{\gamma,n} = \dot{\gamma} + \dot{\mathbf{t}}_\gamma \beta_n + \mathbf{t}_\gamma \dot{\beta}_n$$

Using that $\Phi_{\beta_n}(0) = \Phi_{\beta_n}(1) = (0,1) \in \mathbf{C} \times \mathbf{S}^1$, we deduce that:

$$S_{\gamma,n}(0) = \gamma(0) \quad \text{and} \quad \dot{S}_{\gamma,n}(0) = \big(|\dot\gamma(0)| + |\dot\beta_n(0)|\big)\mathbf{t}_\gamma(0).$$

Similarly, $S_{\gamma,n}(1) = \gamma(1)$ and $\dot S_{\gamma,n}(1)$ is a positive multiple of $\mathbf{t}_\gamma(1)$. This establishes (a).

Let $\phi, \psi\colon [0,1] \to [0,1]$ be as in (29), and set

$$\phi_s(t) = (1-s)\phi(t) + st \quad \text{and} \quad \psi_s(t) = (1-s)\psi(t) + st \quad (s,\, t \in [0,1]). \tag{30}$$

Then

$$(s,t) \mapsto \Phi_\gamma(\phi_s(t))\beta_n(\psi_s(t)) \qquad (s,t \in [0,1])$$

defines a homotopy between B_{γ,n,t_0} and $S_{\gamma,n}$ in $\mathcal{L}^{+\infty}_{-\infty}(P,Q)$. This proves (b).

Part (c) follows from (a) and the fact that the curvatures of α_n, β_n equal $\pm\frac12$ a.e..

Part (d) is a corollary of (5.2).

For part (e), let $H\colon [0,1] \times K \to \mathcal{U}_d$ be a continuous map with $H_0 = f$ and $H_1 = g$. Set

$$\hat H(s,a)(t) = \Phi_{H(s,a)}(\phi(t))\beta_n(\psi(t)) \qquad (s,\, t \in [0,1],\ a \in K).$$

Then $\hat H$ is a homotopy between $B_{f,n,t_0} = \hat H_0$ and $B_{g,n,t_0} = \hat H_1$ in $\mathcal{U}_d$.

Part (f) is obvious. □

(5.6) Lemma. *Let $f\colon K \to \mathcal{C}^{+\infty}_{-\infty}(Q)$ be continuous. Then there exists $n_0 \in \mathbf{N}$ such that $S_{f(a),n} \in \mathcal{L}^{+1}_{-1}(Q)$ for all $a \in K$ whenever $n \ge n_0$ ($n \in \mathbf{N}$).*

Proof. For $a \in K$, let $\gamma_a = f(a)$ and $\mathbf{t}_a = \mathbf{t}_{\gamma_a}$. Let $T = \big\{\frac{1}{2n}, \frac{2}{2n}, \dots, \frac{2n-1}{2n}\big\}$. Then:

$$\begin{aligned}
S_{\gamma_a,n}(t) &= \Phi_{\gamma_a}(t)\beta_n(t) = \gamma_a(t) + \mathbf{t}_a(t)\beta(nt) && (t \in [0,1],\ a \in K)\\
\dot S_{\gamma_a,n}(t) &= \dot\gamma_a(t) + \dot{\mathbf{t}}_a(t)\beta(nt) + n\mathbf{t}_a(t)\dot\beta(nt) && (t \in [0,1],\ a \in K)\\
\ddot S_{\gamma_a,n}(t) &= \ddot\gamma_a(t) + \ddot{\mathbf{t}}_a(t)\beta(nt) + 2n\dot{\mathbf{t}}_a(t)\dot\beta(nt) + n^2\mathbf{t}_a(t)\ddot\beta(nt) && (t \in [0,1] \smallsetminus T,\ a \in K)
\end{aligned}$$

Since $f\colon K \to \mathcal{C}^{+\infty}_{-\infty}(Q)$ is continuous and K compact, $|\gamma_a^{(k)}(t)|$ and $|\mathbf{t}_a^{(k)}(t)|$ $(k = 0,1,2)$ are all bounded by some constant as (t,a) ranges over $[0,1] \times K$. Using expression (11) for the curvature and the multilinearity of the determinant, we conclude that

$$\kappa_{S_{\gamma_a,n}}(t) = \tfrac12 + O(\tfrac1n) \quad (t \in [0,1] \smallsetminus T,\ a \in K),$$

where $O(\frac1n)$ is a function of (t,a) such that $n\big|O(\frac1n)\big|$ is uniformly bounded over $([0,1] \smallsetminus T) \times K$ as n ranges over $\mathbf{N}$. It follows that $S_{\gamma_a,n} \in \mathcal{L}^{+1}_{-1}(Q)$ for all sufficiently large n. □

(5.7) Lemma. *Let $f\colon K \to \mathcal{C}^{+1}_{-1}(Q)$ be continuous, $t_0 \in (0,1)$. Then for all sufficiently large $n \in \mathbf{N}$, there exists a continuous $H\colon [0,1] \times K \to \mathcal{L}^{+1}_{-1}(Q)$ with $H_0 = B_{f,n,t_0}$ and $H_1 = S_{f,n}$.*

Proof. Let H be given by

$$H(s,a)(t) = \Phi_{f(a)}(\phi_s(t))\beta_n(\psi_s(t)) \qquad (s,t \in [0,1],\ a \in K),$$

where ϕ_s, ψ_s are as in (30). Then $H(0,a) = B_{f(a),n,t_0}$ and $H(1,a) = S_{f(a),n}$. A computation entirely similar to the one in the proof of (5.6) establishes that $H(s,a) \in \mathcal{L}^{+1}_{-1}(Q)$ for all $s \in [0,1]$, $a \in K$ if n is sufficiently large. The details will be left to the reader, but to make things easier, notice that ϕ_s, ψ_s are piecewise linear for all $s \in [0,1]$, so that $\ddot\psi_s = \ddot\phi_s = 0$ except at a finite set of points (which depends on s). □

The next result provides a sufficient condition, which does not involve g, for one to be able to write a compact family of curves f as $f = A_{g,n,t_0}$.

(5.8) Lemma. *Let X be a compact Hausdorff topological space and $f\colon X \to \mathcal{U}_d \subset \mathcal{L}^{+1}_{-1}(Q;\theta_1)$, $t_0\colon X \to (0,1)$ be continuous maps. Then it is possible to reparametrize each $f(a)$ (continuously with a) and find a continuous $g\colon X \to \mathcal{L}^{+1}_{-1}(Q;\theta_1)$ so that $f(a) = A_{g(a),n,t_0(a)}$ for all $a \in X$ if and only if there exists a continuous function $\varepsilon\colon X \to (0,1)$ such that, for all $a \in X$:*

(i) $0 < t_0(a) - \varepsilon(a) < t_0(a) + \varepsilon(a) < 1$;

(ii) $f(a)|_{[t_0(a)-\varepsilon(a),\,t_0(a)+\varepsilon(a)]}$ *is some parametrization of* $\Phi_{f(a)}(t_0(a) - \varepsilon(a))\alpha_n$.

Proof. Suppose that such a function $\varepsilon\colon X \to (0,1)$ exists. Since X is compact, we may reparametrize all $f(a)$ so that ε becomes a constant function and, for all $a \in X$, satisfies:

(I) $0 < t_0(a) - 2\varepsilon < t_0(a) + 2\varepsilon < 1$;

(II) $f(a)|_{[t_0(a)-\varepsilon, t_0(a)+\varepsilon]}$ is a parametrization of $\Phi_{f(a)}(t_0(a)-\varepsilon)\alpha_n$ by a multiple of arc-length.

Define $g\colon X \to \mathcal{L}^{+1}_{-1}(Q)$ by:

$$g(a)(t) = \begin{cases} f(a)(t) & \text{if } t \notin [t_0(a)-2\varepsilon, t_0(a)+2\varepsilon] \\ f(a)\big(\frac{1}{2}(t+t_0(a)-2\varepsilon)\big) & \text{if } t \in [t_0(a)-2\varepsilon, t_0(a)] \\ f(a)\big(\frac{1}{2}(t+t_0(a)+2\varepsilon)\big) & \text{if } t \in [t_0(a), t_0(a)+2\varepsilon] \end{cases} \qquad (a \in X,\ t \in [0,1]).$$

Then g is continuous because f and t_0 are continuous, and $f(a) = A_{g(a),n,t_0(a)}$ for all $a \in X$. This proves the "if" part of the lemma. The converse is obvious. □

As a simple application of (5.6), we prove that this article is not a study of the empty set.

(5.9) Lemma. *Let $P = (p,w)$, $Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$ and let $\theta_1 \in \mathbf{R}$ satisfy $e^{i\theta_1} = z\bar{w}$. Then:*

(a) $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q) \neq \emptyset$.

(b) $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1) \neq \emptyset$ *if* $\kappa_1\kappa_2 < 0$.

(c) $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$ *may be empty if* $\kappa_1\kappa_2 \geq 0$.

Proof. By (3.5), we need only consider spaces of the form $\mathcal{L}^{+1}_{-1}(Q)$, $\mathcal{L}^{+\infty}_{0}(Q)$ and $\mathcal{L}^{+\infty}_{1}(Q)$. It is clear that $\mathcal{C}^{+\infty}_{-\infty}(Q) \neq \emptyset$ for all $Q \in UT\mathbf{C}$. Let $\gamma \in \mathcal{C}^{+\infty}_{-\infty}(Q)$ be arbitrary.

By (5.6), if n is sufficiently large, then $S_{\gamma,n} \in \mathcal{L}^{+1}_{-1}(Q)$. Furthermore, attaching loops (possibly with reversed orientation) to $S_{\gamma,n}$, we can obtain a curve in $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ for any $\theta_1 \in \mathbf{R}$ satisfying $e^{i\theta_1} = z$. This proves (b), and also part (a) when $\kappa_1\kappa_2 < 0$.

Similarly, define a curve $\bar{S}_{\gamma,n}$ by $\bar{S}_{\gamma,n}(t) = \Phi_\gamma(t)\big(\frac{1}{4}\alpha_n(t)\big)$ $(t \in [0,1])$. In words, $\bar{S}_{\gamma,n}$ is obtained from γ by spreading n loops of radius $\frac{1}{2}$, instead of n eights of radius 2. Using an argument analogous to the one which established (5.6), one sees that $\bar{S}_{\gamma,n} \in \mathcal{L}^{+\infty}_{1}(Q)$ for all sufficiently large n. This completes the proof of (a).

To see that $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$ may be empty if $\kappa_1\kappa_2 \geq 0$, we use eq. (12): If κ_1, κ_2 are both non-negative, for example, then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ can only contain curves having positive total turning. □

Remark. Invoking (1.15), we obtain a version of (5.9) with $\mathcal{C}$ in place of $\mathcal{L}$.

(5.10) Corollary. *Let $\mathcal{U}_d$ denote the subset of $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ consisting of all diffuse curves, where $Q = (q,z)$ and $e^{i\theta_1} = z$. Then $\mathcal{U}_d \neq \emptyset$.*

Proof. Lemma (5.5 (c)) implies that $B_{\gamma,1,\frac{1}{2}} \in \mathcal{U}_d$ for any $\gamma \in \mathcal{L}^{+1}_{-1}(Q;\theta_1)$. Since the latter is nonempty by (5.9 (b)), so is $\mathcal{U}_d$. □

(5.11) Theorem (Dubins). *Let $x > 0$, $Q = (x,1)$ and $\eta \in \mathcal{L}^{+1}_{-1}(Q)$ be the line segment $\eta\colon t \mapsto xt$. Then η and $B_{\eta,1,\frac{1}{2}}$ lie in the same component of $\mathcal{L}^{+1}_{-1}(Q)$ if and only if $x > 4$.*

Proof. See [3], thm. 5.3. □

The next construction provides a deformation of the straight line segment $[0,x]$ to the same segment with an eight attached which is continuous with respect to x.

(5.12) Construction. For $x > 0$, let $\eta_x\colon [0,x] \to \mathbf{C}$ be the line segment $t \mapsto t$. Take $t_0 = \frac{1}{2}$ in (29) and let $h\colon [0,1] \times [0,6] \to \mathbf{C}$ be a fixed homotopy between $h_0 = \eta_6$ and

$$h_1 = \Phi_{\eta_6}\Big(6\phi\big(\tfrac{t}{6}\big)\Big)\beta_1\big(\psi\big(\tfrac{t}{6}\big)\big) \quad (\eta_6 \text{ with an eight attached})$$

such that $t \mapsto h_s(6t)$ $(t \in [0,1])$ is a curve in $\mathcal{L}^{+1}_{-1}(Q)$ for all $s \in [0,1]$. The existence of h is guaranteed by (5.11). Let $\mu\colon [0,+\infty) \to [0,1]$ be a smooth function such that $\mu(x) = 0$ if $x \in [0,6]$ and $\mu(x) = 1$ if $x \geq 8$. Define a family of curves $\eta^u_x\colon [0,1] \to \mathbf{C}$ by:

$$\eta^u_x(t) = \begin{cases} \eta_x(t) & \text{if } t \geq 6 \text{ or } x \leq 6 \\ h(u\mu(x), t) & \text{if } t \leq 6 \text{ and } x \geq 6 \end{cases} \qquad (u \in [0,1],\ t \in [0,x],\ x > 0). \tag{31}$$

Of course, $\eta_x^0 = \eta_x$ for all $x > 0$. If $x \geq 8$, then η_x^1 equals η_x with an eight attached; in particular, $\eta_x^1|_{[3-6\varepsilon,3]}$ is a loop.

Grafting. We now explain how to graft straight line segments onto a diffuse curve (see fig. 9).

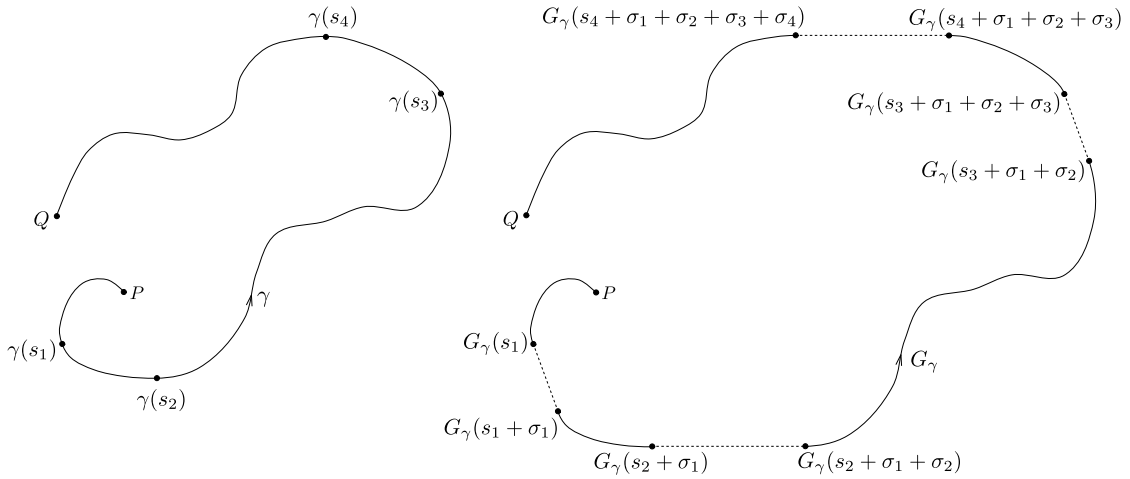
(5.13) Definition. Let $\gamma \in \mathcal{L}_{-1}^{+1}(Q)$ be a curve of length L parametrized by arc-length, $\sigma_i \geq 0$ and $s_i \in [0,1]$, $i = 1, \ldots, 2n$, where the s_i form a non-decreasing sequence. Suppose that there exists a bijection p of $\{1, \ldots, 2n\}$ onto itself such that, for each i:

$(*)$ $\sigma_i = \sigma_{p(i)}$ and $\mathbf{t}_\gamma(s_{p(i)}) = -\mathbf{t}_\gamma(s_i)$.

Then we define the *graft* $G_\gamma = G_{\gamma,(s_i),(\sigma_i)} \colon \big[0, L + \sum_{i=1}^{2n} \sigma_i\big] \to \mathbf{C}$ by:

$$G_\gamma(s) = \begin{cases} \gamma(s) & \text{if } \ s \in [0, s_1] \\ \gamma(s_1) + (s - s_1)\mathbf{t}_\gamma(s_1) & \text{if } \ s \in [s_1, s_1 + \sigma_1] \\ \gamma(s - \sigma_1) + \sigma_1 \mathbf{t}_\gamma(s_1) & \text{if } \ s \in [s_1 + \sigma_1, s_2 + \sigma_1] \\ \gamma(s_2) + \sigma_1 \mathbf{t}_\gamma(s_1) + (s - s_2 - \sigma_1)\mathbf{t}_\gamma(s_2) & \text{if } \ s \in [s_2 + \sigma_1, s_2 + \sigma_1 + \sigma_2] \\ \qquad \vdots & \qquad \vdots \\ \gamma(s - \sum_{i=1}^{2n} \sigma_i) + \sum_{i=1}^{2n} \sigma_i \mathbf{t}_\gamma(s_i) & \text{if } \ s \in [s_{2n} + \sum_{i=1}^{2n} \sigma_i, L + \sum_{i=1}^{2n} \sigma_i] \end{cases} \tag{32}$$

Although it simplifies the previous formula, the assumption that (s_i) is a non-decreasing sequence is not necessary for the construction to work, since we may always relabel the s_i.



(a) (b)

FIGURE 9. A diffuse curve γ and its graft $G_\gamma = G_{\gamma,(s_1,s_2,s_3,s_4),(\sigma_1,\sigma_2,\sigma_3,\sigma_4)}$.

(5.14) Lemma. *Let $\gamma \in \mathcal{L}_{-1}^{+1}(Q)$ be diffuse and G_γ be as in (5.13). Then G_γ is parametrized by arc-length and it lies in the same connected component of $\mathcal{L}_{-1}^{+1}(Q)$ as γ.*

Proof. It is obvious from (32) that $\Phi_{G_\gamma}(0) = \Phi_\gamma(0)$. Looking at the last line of (32) and using $(*)$ we deduce that

$$G_\gamma(s) = \gamma\Big(s - \textstyle\sum_{i=1}^{2n} \sigma_i\Big) \ \text{ for } \ s \in \big[s_{2n} + \sum_{i=1}^{2n} \sigma_i, L + \sum_{i=1}^{2n} \sigma_i\big].$$

Hence, $\Phi_{G_\gamma}(L + \sum_{i=1}^{2n} \sigma_i) = \Phi_\gamma(L)$. Since G_γ is made up of line segments and arcs of γ (composed with translations), $G_\gamma \in \mathcal{L}_{-1}^{+1}(Q)$. It is clear that G_γ is parametrized by arc-length. Finally,

$$u \mapsto G_{\gamma,(s_i),(u\sigma_i)} \qquad (u \in [0,1])$$

defines a path in $\mathcal{L}_{-1}^{+1}(Q)$ joining γ to G_γ. □

Contractibility of $\mathcal{U}_d$. Recall that K denotes a compact manifold, possibly with boundary.

(5.15) Lemma. *Let $f\colon K \to \mathcal{U}_d$ be continuous. Then there exist an open cover $(V_j)_{j=1}^m$ of K and continuous maps $\tau_j^{\pm}\colon K \to (0,1)$, $f_1\colon K \to \mathcal{U}_d$ such that:*

(i) *$f \simeq f_1$ within $\mathcal{U}_d$ and f_1 satisfies conditions (ii) and (iii) of (1.15).*
(ii) *$\mathbf{t}_{f_1(a)}(\tau_j^+(a)) = -\mathbf{t}_{f_1(a)}(\tau_j^-(a))$ whenever $a \in V_j$.*

Proof. Apply (1.15) to f and $\mathcal{U}_d$ to obtain f_1. The idea is to use the implicit function theorem to find $\tau_j^{\pm}$. However, some care must be taken since f_1 need not be differentiable with respect to a.

For each $a \in K$, let $\theta_a\colon [0,1] \to \mathbf{R}$ be the argument of $\mathbf{t}_{f_1(a)}$ satisfying $\theta_a(0) = 0$, and set

$$\bar\theta_a = \frac{1}{2}\Big(\max_{t\in[0,1]} \theta_a(t) + \min_{t\in[0,1]} \theta_a(t)\Big).$$

Because each γ_a is diffuse and K is compact, we can find $\delta > 0$ such that

$$\theta_a([0,1]) \supset \Big(\bar\theta_a - \frac{\pi}{2} - \delta\,,\, \bar\theta_a + \frac{\pi}{2} + \delta\Big) \quad \text{for all } a \in K.$$

Fix $a_0 \in K$. By Sard's theorem, we can find $\psi \in (\bar\theta_{a_0} + \frac{\pi}{2}, \bar\theta_{a_0} + \frac{\pi}{2} + \delta)$ such that both ψ and $\psi - \pi$ are regular values of θ_{a_0}. Let $\tau^{\pm}(a_0) \in (0,1)$ satisfy $\theta_{a_0}(\tau^+(a_0)) = \psi$ and $\theta_{a_0}(\tau^-(a_0)) = \psi - \pi$. No generality is lost in assuming that $\dot\theta_{a_0}(\tau^+(a_0)) > 0$. From eq. (12), $\dot\theta_a = \big|\dot\gamma_{f_1(a)}\big|\,\kappa_{f_1(a)}$. Thus, $\dot\theta_a$ depends continuously on a, so we can find $\mu, \varepsilon > 0$ and a compact neighborhood $V \subset K$ of a_0 such that

$$\psi \in \theta_a((\tau^+(a_0) - \varepsilon, \tau^+(a_0) + \varepsilon)) \;\text{ and }\; \dot\theta_a(t) > \mu \;\text{ whenever } a \in V,\ \big|t - \tau^+(a_0)\big| < \varepsilon.$$

Hence, for each $a \in V$, there exists a *unique* $\tau^+(a) \in (\tau^+(a_0) - \varepsilon, \tau^+(a_0) + \varepsilon)$ with $\theta_a(\tau^+(a)) = \psi$. We claim that the function $\tau^+\colon V \to (0,1)$ so defined is continuous. Consider the equation

$$\theta_a(\tau^+(b)) - \theta_a(\tau^+(a)) = \big[\theta_b(\tau^+(b)) - \theta_a(\tau^+(a))\big] + \big[\theta_a(\tau^+(b)) - \theta_b(\tau^+(b))\big] \quad (a, b \in V).$$

The first term on the right side equals 0 by the definition of τ^+, and the second converges to 0 as $b \to a$ since $\theta_b(t)$ is a uniformly continuous function of $(b,t) \in K \times [0,1]$. Hence, by the mean-value theorem,

$$\big|\tau^+(b) - \tau^+(a)\big| < \frac{1}{\mu}\big|\theta_a(\tau^+(b)) - \theta_a(\tau^+(a))\big| \to 0 \;\text{ as }\; b \to a \quad (a, b \in V).$$

It follows that τ^+ is continuous. Similarly, reducing V if necessary, we can find a continuous function $\tau^-\colon V \to (0,1)$ with $\theta_a(\tau^-(a)) = \psi - \pi$ for all $a \in V$. To finish the proof, cover K by finitely many such compact neighborhoods V_j, let $\tau_j^{\pm}\colon V_j \to (0,1)$ be the corresponding functions and extend each $\tau_j^{\pm}$ to K using the Tietze extension theorem. □

(5.16) Lemma. *Let $f\colon K \to \mathcal{U}_d$ be continuous. Then there exist an open cover $(W_j)_{j=1}^m$ of K and continuous maps $t_j\colon K \to (0,1)$, $g_j\colon W_j \to \mathcal{L}_{-1}^{+1}(Q)$ and $f_2\colon K \to \mathcal{U}_d$ such that:*

(i) *$f \simeq f_2$ within $\mathcal{U}_d$;*
(ii) *$f_2(a) = A_{g_j(a),1,t_j(a)}$ for all $a \in W_j$.*

Proof. Take f_1 as in (5.15). By (1.18), we may assume that each $\gamma_a = f_1(a)\colon [0, L_a] \to \mathbf{C}$ is parametrized by arc-length, so that now $\tau_j^{\pm}(a) \in (0, L_a)$ for each a. Let $(\lambda_j)_{j=1}^m$ be a partition of unity subordinate to $(V_j)_{j=1}^m$, with V_j as in (5.15). Set $\sigma_j = 10m\lambda_j$ and $W_j = \big\{a \in K : \sigma_j(a) > 8\big\}$. Then $\overline{W}_j \subset V_j$ and the W_j form an open cover of K. Define

$$\gamma_a^u = G_{\gamma_a, (\tau_1^-(a), \ldots, \tau_m^-(a), \tau_1^+(a), \ldots, \tau_m^+(a)), (u\sigma_1(a), \ldots, u\sigma_m(a), u\sigma_1(a), \ldots, u\sigma_m(a))} \quad (u \in [0,1],\ a \in K),$$

as in (5.13). Let us suppose that $\tau_1^- \le \cdots \le \tau_m^-(a) \le \tau_1^+(a) \le \cdots \le \tau_m^+(a)$ for each a to abbreviate the notation, and set

$$\xi_j^-(a) = \sum_{i<j} \sigma_i(a) \;\text{ and }\; \xi_j^+(a) = 10m + \sum_{i<j} \sigma_i(a) \quad (a \in K,\ j = 1, \ldots, m).$$

Then

$$\gamma_a^1\big([\tau_j^-(a) + \xi_j^-(a), \tau_j^-(a) + \xi_j^-(a) + \sigma_j(a)]\big)$$

is a line segment, corresponding to the graft at $\gamma_a(\tau_j^-(a))$. Its length $\sigma_j(a)$ is at least 8 if $a \in \overline{W}_j$. Of course, the same statements hold with $^+$ instead of $^-$. We obtain f_2 by deforming all of these segments to eights. More precisely, for $u \in [1,2]$ and $a \in K$, let

$$\gamma_a^u(s) = \begin{cases} \Phi_{\gamma_a^1}(\tau_j^\pm(a) + \xi_j^\pm(a))\eta^{u-1}_{\sigma_j(a)}(s - \tau_j^\pm(a) - \xi_j^\pm(a)) \\ \gamma_a^1(s) \end{cases} \qquad (s \in [0, L_a + 20m])$$

according as $s \in [\tau_j^\pm(a) + \xi_j^\pm(a)\,,\, \tau_j^\pm(a) + \xi_j^\pm(a) + \sigma_j(a)]$ for some j or not, respectively. Here η_x^u is as in (5.12). Let $f_2\colon K \to \mathcal{U}_d$ be given by $f_2(a) = \gamma_a^2$. Note that

$$\gamma_a^2\big([\tau_j^\pm(a) + \xi_j^\pm(a) + 3 - 6\varepsilon\,,\, \tau_j^\pm(a) + \xi_j^\pm(a) + 3]\big) \qquad (j = 1, \dots, m)$$

is a loop whenever $a \in \overline{W}_j$. Thus (after reparametrizing the γ_a^2 so that their domains become $[0,1]$) we may apply (5.8) to each family $f_2|_{\overline{W}_j}$ to find $g_j\colon \overline{W}_j \to \mathcal{L}^{+1}_{-1}(Q)$ and $t_j\colon \overline{W}_j \to (0,1)$ such that

$$f_2(a) = A_{g_j(a),1,t_j(a)} \quad \text{for all } a \in W_j.$$

The functions t_j may be extended to all of K by the Tietze extension theorem. □

(5.17) Lemma. *Let $f\colon K \to \mathcal{U}_d$ be continuous. Suppose that there exists a covering of K by open sets W_j and continuous maps $t_j\colon K \to (0,1)$, $g_j\colon W_j \to \mathcal{L}^{+1}_{-1}(Q)$ with $f(a) = A_{g_j(a),1,t_j(a)}$ whenever $a \in W_j$, $j = 1, \dots, m$. Then there exist continuous $g\colon K \to \mathcal{L}^{+1}_{-1}(Q)$ and $H\colon [0,1] \times K \to \mathcal{U}_d$ with $H_0 = f$ and $H_1 = A_{g,1,\frac{1}{2}}$.*

Proof. The proof will be by induction on m. If $m = 1$ then $W_1 = K$, and H just slides the loop from t_1 to $\frac{1}{2}$:

$$H(s,a) = A_{g_1(a),1,\,(1-s)t_1(a)+\frac{s}{2}} \quad (s \in [0,1],\ a \in K).$$

Suppose now that $m > 1$. Let W be an open set such that $\overline{W} \subset W_m$ and $W_1 \cup \dots W_{m-1} \cup W = K$. Let $\lambda\colon K \to [0,1]$ be a continuous function such that $\lambda(a) = 1$ if $a \in W$ and $\lambda(a) = 0$ if $a \notin W_m$. Define $\hat{H}\colon [0,1] \times K \to \mathcal{U}_d$ by:

$$\hat{H}(s,a) = \begin{cases} A_{g_m(a),1,\,(1-\lambda(a)s)t_m(a)+\lambda(a)st_{m-1}(a)} & \text{if } a \in W_m \\ f(a) & \text{if } a \notin W_m \end{cases} \qquad (s \in [0,1],\ a \in K).$$

Then the induction hypothesis applies to $\hat{H}_1\colon K \to \mathcal{U}_d$, the open sets $\hat{W}_i = W_i$ $(i = 1, \dots, m-2)$ and $\hat{W}_{m-1} = W_{m-1} \cup W$, and the same functions t_j as before, $j = 1, \dots, m-1$. The existence of $\hat{g}_{m-1}$ as in the statement is guaranteed by (5.8): Using (5.5(f)), we deduce that there is at least one loop at $t_{m-1}(a)$ for $a \in \hat{W}_{m-1}$. □

(5.18) Proposition. *Let $f\colon K \to \mathcal{U}_d$ be continuous. Then $f \simeq B_{f,n,\frac{1}{2}}$ within $\mathcal{U}_d$ for all $n \geq 1$.*

Proof. Applying (5.16) and (5.17) to f, we obtain continuous maps $g\colon K \to \mathcal{L}^{+1}_{-1}(Q)$ and $h\colon K \to \mathcal{U}_d$ such that $f \simeq h$ in $\mathcal{U}_d$ and

$$h(a) = A_{g(a),1,\frac{1}{2}} \quad \text{for all } a \in K.$$

Using (5.5(d)), we may deform the loop at $t = \frac{1}{2}$ to attach n eights to h at $t = \frac{1}{2}$ (for arbitrary $n \geq 1$). Thus $h \simeq B_{h,n,\frac{1}{2}}$. Together with (5.5(e)), this implies that $f \simeq B_{f,n,\frac{1}{2}}$ within $\mathcal{U}_d$. □

(5.19) Theorem C. *Let $Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$ and $\theta_1 \in \mathbf{R}$ satisfy $e^{i\theta_1} = z$. Then the subspace $\mathcal{U}_d \subset \mathcal{L}^{+1}_{-1}(Q;\theta_1)$ consisting of all diffuse curves is homeomorphic to $\mathbf{E}$, hence contractible.*

Proof. Because $\mathcal{U}_d$ is open, it suffices to prove that it is weakly contractible, by (1.12(b)). Let $k \in \mathbf{N}$, $f\colon \mathbf{S}^k \to \mathcal{U}_d$ be continuous and $g\colon \mathbf{S}^k \to \mathcal{U}_d$ be a map satisfying (i)–(iii) of (1.15) (with $\mathcal{U} = \mathcal{U}_d$). By (5.1), there exists $G\colon [0,1] \times \mathbf{S}^k \to \mathcal{C}^{+\infty}_{-\infty}(Q)$ such that $G_0 = g$ and G_1 is a constant map. By (5.6), there exists $n_0 \in \mathbf{N}$ such that if $n \geq n_0$, then $S_{G(s,a),n} \in \mathcal{U}_d$ for all $s \in [0,1]$, $a \in \mathbf{S}^k$. Applying (5.18) and (5.7) to g, we obtain $n_1 \geq n_0$ and a continuous $F\colon [-1,0] \times \mathbf{S}^k \to \mathcal{U}_d$ with $F_{-1} = g$ and $F_0 = S_{g,n_1}$. Concatenating F and S_{G,n_1} we obtain a null-homotopy of g in $\mathcal{U}_d$. □

6. Critical curves

Fix $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$ and $\theta_1 \in \mathbf{R}$ satisfying $e^{i\theta_1} = z$. Let $\gamma \in \mathcal{L}^{+1}_{-1}(Q; \theta_1)$ and $\theta \colon [0,1] \to \mathbf{R}$ be the argument of $\mathbf{t}_\gamma$ satisfying $\theta(0) = 0$. Finally, let

$$\theta^+ = \max_{t \in [0,1]} \theta(t) \quad \text{and} \quad \theta^- = \min_{t \in [0,1]} \theta(t). \tag{33}$$

Recall that γ is called *critical* if $\theta^+ - \theta^- = \pi$. A curve $\eta \in \mathcal{L}^{+1}_{-1}(Q; \theta_1)$ must be either condensed, diffuse or critical. It has already been shown that the subspaces $\mathcal{U}_c$ and $\mathcal{U}_d$ of $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ consisting of all condensed (resp. diffuse) curves are both contractible. Let $\mathcal{T} \subset \mathcal{L}^{+1}_{-1}(Q; \theta_1)$ denote the subspace of all critical curves. Clearly, $\mathcal{T}$ is closed as the complement of $\mathcal{U}_c \cup \mathcal{U}_d$. Since the difference $\theta^+ - \theta^-$ depends continuously on γ, we deduce that $\partial\mathcal{U}_c \subset \mathcal{T}$ and $\partial\mathcal{U}_d \subset \mathcal{T}$, where $\partial\mathcal{U}_c$ denotes the boundary of $\mathcal{U}_c$ considered as a subspace of $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ and similarly for $\mathcal{U}_d$.

(6.1) Proposition. *Let $|\theta_1| < \pi$ and $\mathcal{U}_c$, $\mathcal{U}_d$, $\mathcal{T} \subset \mathcal{L}^{+1}_{-1}(Q; \theta_1)$ be as above. Then $\partial\mathcal{U}_c = \partial\mathcal{U}_d = \mathcal{T}$. Therefore, $\bar{\mathcal{U}}_c \cup \bar{\mathcal{U}}_d = \mathcal{L}^{+1}_{-1}(Q; \theta_1)$ and $\bar{\mathcal{U}}_c \cap \bar{\mathcal{U}}_d = \mathcal{T}$.*

Observe that $\mathcal{T} = \emptyset$ if $|\theta_1| > \pi$ and $\mathcal{U}_c = \emptyset$ if $|\theta_1| \geq \pi$. If $|\theta_1| = \pi$, it is still true that $\partial\mathcal{U}_d = \mathcal{T}$.

Proof. Let $\gamma \in \mathcal{L}^{+1}_{-1}(Q; \theta_1)$ be a critical curve and $\mathcal{V} \subset \mathcal{L}^{+1}_{-1}(Q; \theta_1)$ be an open set containing γ. Let θ be the argument of $\mathbf{t}_\gamma$ satisfying $\theta(0) = 0$, and let θ^+, θ^- be as in (33).

We first prove that $\mathcal{V} \cap \mathcal{U}_c \neq \emptyset$. Our immediate objective is to replace γ with another curve in $\mathcal{V} \cap \mathcal{T}$ having smaller curvature. Choose $t_1 \in (0,1)$ and $\delta > 0$ such that $\theta(t) \in (\theta^-, \theta^+)$ for all $t \in [t_1 - \delta, t_1]$. Let $Q_0 = \Phi_\gamma(t_1 - \delta)$, $Q_1 = \Phi_\gamma(t_1)$ and consider the map

$$F \colon \mathcal{L}^{+1}_{-1}(Q_0, \cdot) \to UT\mathbf{C}, \quad F(\eta) = \Phi_\eta(1).$$

(Recall that $\mathcal{L}^{+1}_{-1}(Q_0, \cdot)$ consists of all $(-1,1)$-admissible curves having initial frame equal to Q_0 and arbitrary final frame.) By (1.6), F is an open map. It follows that for any $\tilde{Q}_1$ close enough to Q_1, we can find $\eta \in \mathcal{L}^{+1}_{-1}(Q_0, \tilde{Q}_1)$ such that

$$\theta_\eta([0,1]) \subset (\theta^-, \theta^+). \tag{34}$$

Let $Q_1 = (q_1, z_1)$ and $Q = (q, z)$. Since γ is critical, the image of $\mathbf{t}_\gamma$ is contained in a semicircle. Consequently, $q \neq 0$. Choose $\kappa_0 \in (0,1)$ close to 1. Replace the arc $\gamma|_{[t_1-\delta, t_1]}$ by a curve η as above with $\tilde{Q}_1 = (q_1 + (\kappa_0 - 1)q, z_1)$, and the arc $\gamma|_{[t_1,1]}$ by its translate $\gamma|_{[t_1,1]} + (\kappa_0 - 1)q$. Let γ_1 be the resulting curve; observe that γ_1 is critical, $\Phi_{\gamma_1}(0) = (0,1)$ and $\Phi_{\gamma_1}(1) = (\kappa_0 q, z)$. Set $\gamma_2 = \frac{1}{\kappa_0}\gamma_1$ (that is, $\gamma_2(t)$ is obtained from $\gamma_1(t)$ by a dilatation through a factor of $\frac{1}{\kappa_0}$ for all $t \in [0,1]$). Then

$$\Phi_{\gamma_2}(0) = (0,1), \quad \Phi_{\gamma_2}(1) = (q, z), \quad \mathbf{t}_{\gamma_2}(t) = \mathbf{t}_{\gamma_1}(t) \ \text{ and } \ \kappa_{\gamma_2}(t) = \kappa_0 \kappa_{\gamma_1}(t) \ \text{ for all } t \in [0,1].$$

Thus, γ_2 is a critical curve in $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ whose curvature is constrained to $(-\kappa_0, \kappa_0)$. Moreover, if κ_0 is close enough to 1 and η is chosen appropriately, we can guarantee that $\gamma_2 \in \mathcal{V}$.

Having established the existence of γ_2 with these properties, let us return to the beginning, setting $\gamma = \gamma_2 \in \mathcal{V}$. Since $|\theta_1| < \pi$, either

$$\theta^{-1}(\{\theta_-\}) \cap \{0,1\} = \emptyset \quad \text{or} \quad \theta^{-1}(\{\theta_+\}) \cap \{0,1\} = \emptyset,$$

and we lose no generality in assuming the latter. Choose $\varepsilon > 0$ small enough to guarantee that

$$W = \theta^{-1}\big((\theta_+ - \varepsilon, \theta_+]\big) \subset (0,1).$$

Cover $\theta^{-1}(\{\theta_+\})$ by the finite union of disjoint intervals $(a_i, b_i) \subset W$ with $\theta(a_i) = \theta(b_i) = \theta^+ - \varepsilon$, $i = 1, \ldots, m$. Let $P_i = \Phi_\gamma(a_i)$, $Q_i = \Phi_\gamma(b_i)$. We can obtain a curve in $\mathcal{U}_c \cap \mathcal{V}$ by modifying γ in each of these intervals to avoid the argument θ_+ using (4.4): Note that $P_i^{-1}\gamma|_{[a_i,b_i]}$ satisfies the hypotheses of (4.15) because it has curvature in the open interval $(-\kappa_0, +\kappa_0)$ and is not a line segment. Moreover, the inclusion $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(P_i^{-1}Q_i) \to \mathcal{L}^{+1}_{-1}(P_i^{-1}Q_i)$ is continuous by (2.4).

The proof that $\mathcal{V} \cap \mathcal{U}_d \neq \emptyset$ is easier. Let the critical curve $\gamma \colon [0, L] \to \mathbf{C}$ be parametrized by arc-length. Then we can find s_0, $s_1 \in [0,1]$ with $\mathbf{t}_\gamma(s_0) = -\mathbf{t}_\gamma(s_1)$. Choose $\varepsilon > 0$ and let

$$G_\gamma = G_{\gamma,(s_0,s_1),(\varepsilon,\varepsilon)}.$$

(See definition (5.13) and fig. 9.) Choose $\kappa_0 \in (0,1)$ and construct a curve $\zeta \in \mathcal{L}^{+1}_{-1}(Q;\theta_1)$ by replacing the line segment $G_\gamma|_{[s_0,s_0+\varepsilon]}$ by three small arcs of circles of radius $\frac{1}{\kappa_0}$ as indicated in fig. 10. If the bump is chosen to lie on the correct side, the curve ζ will be diffuse, and if $\varepsilon > 0$ is small enough, then $\zeta \in \mathcal{V}$. (Notice that this part of the proof works even if $|\theta_1| = \pi$.)

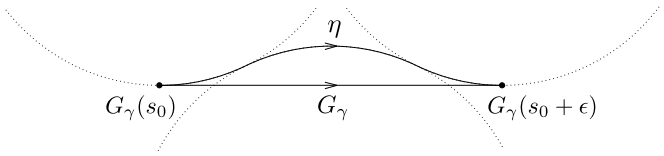


FIGURE 10.

We have established that $\mathcal{T} \subset \partial\mathcal{U}_d \cap \partial\mathcal{U}_c$. As explained at the beginning of the section, $\partial\mathcal{U}_c \subset \mathcal{T}$ and $\partial\mathcal{U}_d \subset \mathcal{T}$. Thus, $\partial\mathcal{U}_c = \partial\mathcal{U}_d = \mathcal{T}$. □

Existence of critical curves. It is immediate from the definition of critical that if $|\theta_1| > \pi$, then the subspace $\mathcal{T} \subset \mathcal{L}^{+1}_{-1}(Q;\theta_1)$ consisting of all critical curves must be empty. In this subsection we shall determine exactly when $\mathcal{T} = \emptyset$ for $|\theta_1| \le \pi$.

(6.2) Definition. Let σ be *sign sequence*, that is, an alternating finite sequence of signs, such as $+-+$ or $-+-+$. Let $|\sigma| \ge 2$ denote the number of terms in the sequence, and let σ_i denote the i-th term of σ $(1 \le i \le |\sigma|)$. The *opposite* $-\sigma$ of σ is the unique sign sequence satisfying $|-\sigma| = |\sigma|$ and $(-\sigma)_i = -\sigma_i$. A critical curve $\gamma\colon [0,1] \to \mathbf{C}$ is *of type* σ if there exist $0 \le t_1 < t_2 < \cdots < t_m \le 1$ with $\theta(t_i) = \theta^{\sigma_i}$. (Recall that $\theta^+ = \max_{t\in[0,1]} \theta(t)$ and $\theta^- = \min_{t\in[0,1]} \theta(t)$, where $e^{i\theta} = \mathbf{t}_\gamma$.)

Given a sign sequence σ, one can determine whether there exist critical curves of type σ in $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ using an elementary geometric construction, see fig. 11.

(6.3) Proposition. *Let $\theta_1 \in [0,\pi]$, $z = e^{i\theta_1}$ and $Q = (q,z) \in \mathbf{C}\times\mathbf{S}^1$. Let σ be a sign sequence,*

$$a = i\sigma_1\big(1 + (-1)^{|\sigma|+1} z\big) \in \mathbf{C} \quad \text{and} \quad r = 2\,|\sigma| \in \mathbf{N}.$$

Let R_σ be the open region of the plane which does not *contain the negative real axis and which is bounded by the shortest arc of $C_r(a)$ joining $a+ri$ to $a-riz$ and the tangent lines to $C_r(a)$ at these points. Then $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ contains critical curves of type σ if and only if $q \in R_\sigma$.*

We have assumed that $\theta_1 \in [0,\pi]$ just to simplify the statement. If $\theta_1 \in [-\pi,0]$, the only difference is that the points bounding the arc of $C_r(a)$ are now $a-ri$ and $a+riz$. Indeed, reflection across the x-axis yields a homeomorphism between $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ and $\mathcal{L}^{+1}_{-1}(\bar{Q};-\theta_1)$, where $\bar{Q} = (\bar{q},\bar{z})$, which maps critical curves of type σ to critical curves of type $-\sigma$.

When $\theta_1 = 0$, the points $a+ri$ and $a-ri$ determine two shortest arcs of $C_r(a)$, not just one; the region R_σ is bounded by the one which goes through $a+r$. When $\theta_1 = \pm\pi$, the arc of circle degenerates to a single point. In this case, R_σ is bounded by the horizontal line through $a \pm ri$.

Proof of (6.3). There are four essentially distinct types of sign sequences to consider:

$$\underbrace{+-\cdots+-}_{2n}\ ,\ \underbrace{-+\cdots-+}_{2n}\ ,\ +\underbrace{-+\cdots-+}_{2n} \text{ and } -\underbrace{+-\cdots+-}_{2n} \quad (n \in \mathbf{N},\ n \ge 1).$$

(Note that these are distinguished by the values of σ_1 and $|\sigma|$ appearing in the expression for a.) We shall prove the theorem for a sequence of the first type; the proof in the remaining three cases is analogous. The argument given here is the same as the one which was used to prove (4.16), so some details will be omitted.

For each $\mu \in [\theta_1 - \pi, 0]$, let $\gamma_\mu\colon [0, 2n\pi + \theta_1] \to \mathbf{C}$ be the unique curve parametrized by arc-length satisfying:

$$\gamma_\mu(0) = 0 \text{ and } \mathbf{t}_{\gamma_\mu}(s) = \begin{cases} e^{is} & \text{if } s \in [0,\mu+\pi] \cup [\mu+2n\pi, \theta_1+2n\pi] \bigcup_k [\mu+k\pi, \mu+(k+1)\pi] \\ e^{i(2\mu-s)} & \text{if } s \in \bigcup_k [\mu+k\pi, \mu+(k+1)\pi] \end{cases}$$

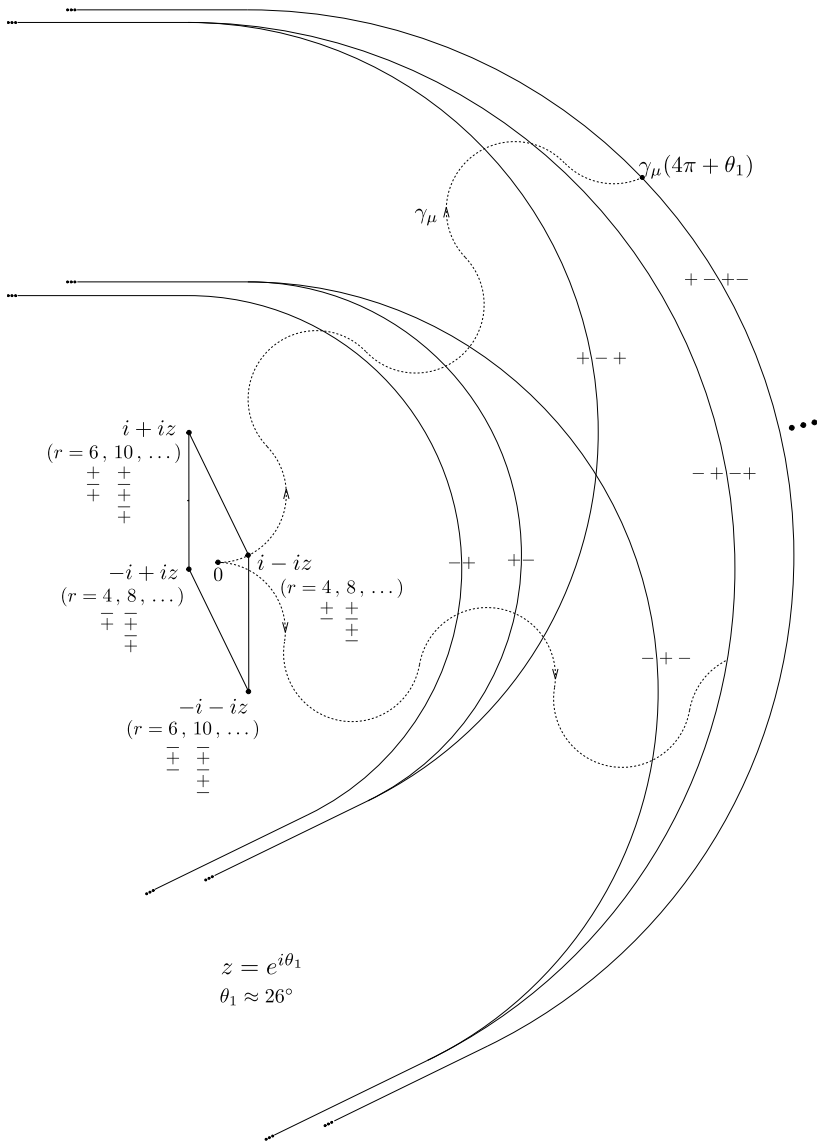


FIGURE 11. The regions R_σ of (6.3).

where the first (resp. second) union is over all $k \equiv 0$ (resp. $k \equiv 1$) (mod 2), $1 \le k \le 2n-1$. Notice that γ_μ is the concatenation of arcs of circles of radius 1, see fig. 11. (Vaguely speaking, γ_μ is the "most efficient" critical curve γ of type σ with $\min\theta_\gamma = \mu$ and $|\kappa_\gamma| \le 1$.) We have:

$$\Phi_{\gamma_\mu}(0) = (0,1), \quad \mathbf{t}_{\gamma_\mu}(2n\pi+\theta_1) = z, \quad \min\theta_{\gamma_\mu} = \mu, \quad \max\theta_{\gamma_\mu} = \mu+\pi \quad \text{and}$$
$$\gamma_\mu(2n\pi+\theta_1) = \left(\int_0^{\mu+\pi} + (2n-1)\int_\mu^{\mu+\pi} + \int_\mu^{\theta_1}\right) e^{is}\,ds = (i-iz) + 4nie^{i\mu}.$$

From the previous equation it follows that as μ increases from $\theta_1-\pi$ to 0, the endpoint of γ_μ traces out the arc of $C_r(a)$ joining $a-riz$ to $a+ri$, where $a = i-iz$ and $r = 4n = 2\,|\sigma|$. Further, the tangent line to $C_r(a)$ at $\gamma_\mu(2n\pi+\theta_1)$ is parallel to $e^{i\mu}$, for it must be orthogonal to $4nie^{i\mu}$.

It is easy to see that any $q \in \overline{R}_\sigma$ is the endpoint of a curve of one of the following three types:

(i) The concatenation of γ_μ with a line segment of direction z, for some $\mu \in [\theta_1-\pi, 0]$.
(ii) The concatenation of $\gamma_0|_{[0,\pi]}$, a line segment of length $\ell \ge 0$ having direction -1, the arc $-\ell+\gamma_0|_{[\pi,2n\pi+\theta_1]}$, and a line segment of direction z.

(iii) The concatenation of $\gamma_{\theta_1-\pi}|_{[0,\theta_1+\pi]}$, a line segment of length $\ell_1 \geq 0$ of direction $-z$, the arc $-\ell_1 z + \gamma_0|_{[\theta_1+\pi,\theta_1+\frac{3\pi}{2}]}$, a line segment of length $\ell_2 \geq 0$ and direction $-iz$, the arc $-\ell_1 z - \ell_2 iz + \gamma_0|_{[\theta_1+\frac{3\pi}{2},2n\pi+\theta_1]}$, and a line segment of direction z.

If $q \in R_\sigma$, then we can find a critical curve γ of type σ in $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ by a slight modification of one of these curves.

Conversely, suppose that $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ contains critical curves of type σ. Let $\eta\colon [0,L] \to \mathbf{C}$ be such a curve, parametrized by arc-length, and let $\mu = \min\theta$, where $\theta\colon [0,L] \to \mathbf{R}$ is the argument of $\mathbf{t}_\eta$ satisfying $\theta(0) = 0$. Define

$$g\colon [0,L] \to \mathbf{R} \quad \text{by} \quad g(s) = \big\langle \eta(s) - \gamma_\mu(2n\pi + \theta_1)\,,\, ie^{i\mu} \big\rangle.$$

Notice that $g(s) > 0$ if and only if $\eta(s)$ lies to the left of the line through $\gamma_\mu(2n\pi+\theta_1) \in C_r(a)$ having direction $e^{i\mu}$; we have already seen that this line is tangent to $C_r(a)$ at this point. We claim that $g(L) > 0$. Since η is critical, $\theta(s) \in [\mu, \mu+\pi]$ for all s. Hence,

$$g'(s) = \big\langle e^{i\theta(s)}, ie^{i\mu} \big\rangle = \cos\big(\theta(s) - (\mu + \tfrac{\pi}{2})\big) \geq 0 \quad \text{for all } s \in [0,L]. \tag{35}$$

Let $J_i = (a_i, b_i) \subset (0,L)$, $i = 0, \ldots, 2n = |\sigma|$, be disjoint intervals such that:

(I) $\theta(a_0) = 0$ and $\theta(a_1) = \mu + \pi$;
(II) $\theta(a_i) = \mu + \pi$ and $\theta(b_i) = \mu$ for $i = 1, 3, \ldots, 2n-1$;
(III) $\theta(a_i) = \mu$ and $\theta(b_i) = \mu + \pi$ for $i = 2, 4, \ldots, 2n-2$;
(IV) $\theta(a_{2n}) = \mu + \pi$ and $\theta(b_{2n}) = \theta_1$.

Such intervals exist because $\theta([0,L]) \subset [\mu, \mu+\pi]$, $\theta(0) = 0$, $\theta(L) = \theta_1$ and η is critical of type σ. It follows from (35) and the fact θ is strictly 1-Lipschitz (cf. (2.2) and eq. (12)) that

$$\begin{aligned} g(L) - g(0) &\geq \left(\sum_{i=0}^{2n} \int_{a_i}^{b_i} \right) g'(s)\, ds \\ &> \left(\int_0^{\mu+\pi} + (2n-1) \int_\mu^{\mu+\pi} + \int_\mu^{\theta_1} \right) \big\langle e^{it}, ie^{i\mu} \big\rangle \, dt = \big\langle \gamma_\mu(2n\pi+\theta_1), ie^{i\mu} \big\rangle. \end{aligned}$$

Therefore, $g(L) > 0$ as claimed. We conclude that $q = \eta(L)$ lies on the side of a tangent to $C_r(a)$ which only contains points of R_σ. □

(6.4) Corollary. *Let $Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$,*

$$a = \operatorname{sign}(\operatorname{Im}(z))\, i(z-1) \in \mathbf{C}$$

and let $R_{\mathcal{T}}$ be the open region of the plane which does not *contain the negative real axis and which is bounded by the shortest arc of $C_4(a)$ joining $a + \operatorname{sign}(\operatorname{Im}(z))\, 4i$ to $a - \operatorname{sign}(\operatorname{Im}(z))\, 4iz$ and the tangent lines to $C_4(a)$ at these points. Then $\mathcal{L}^{+1}_{-1}(Q)$ contains critical curves if and only if $q \in R_{\mathcal{T}}$.*

Proof. Let $z = e^{i\theta_1}$, $|\theta_1| \leq \pi$. For $\theta_1 \in [0,\pi]$ (resp. $\theta_1 \in [-\pi, 0]$), $R_{\mathcal{T}}$ is the same as the region R_{-+} (resp. R_{+-}) appearing in proposition (6.3). □

If $z = \pm 1$, then $\operatorname{sign}(\operatorname{Im} z)$ is not defined. When $z = 1$, $\mathcal{R}_{\mathcal{T}}$ is bounded by the semicircle centered at 0 through 4 and $\pm 4i$ and the tangents to $C_4(0)$ at the latter two points. When $z = -1$, $R_{\mathcal{T}}$ is bounded by the horizontal lines through $\pm 2i$.

(6.5) Corollary. *Let $Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$ and $e^{i\theta_1} = z$. Then there exist condensed curves but not critical curves in $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ if and only if $|\theta_1| < \pi$ and q lies in the region illustrated in fig. 2.*

Proof. This is an immediate consequence of (4.16) and (6.4). □

(6.6) Lemma. *Let $Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$ and $e^{i\theta_1} = z$, $|\theta_1| \leq \pi$. Let $\omega \in [|\theta_1|, \pi]$ and $r(\omega) = 4\sin\left(\frac{\omega}{2}\right)$. Suppose that q lies inside of $C_{r(\omega)}\big(\operatorname{sign}(\theta_1) i(z-1)\big)$. Then there does not exist a curve in $\hat{\mathcal{L}}^{+1}_{-1}(Q;\theta_1)$ or $\mathcal{L}^{+1}_{-1}(Q;\theta_1)$ having amplitude ω.*

Proof. Assume that $\theta_1 \in [0, \pi]$; the proof for $\theta_1 \in [-\pi, 0]$ is analogous. Let $\omega \in [\theta_1, \pi]$, $\mu \in [\theta_1 - \omega, 0]$ and let $\gamma_\mu \colon [0, 2\omega - \theta_1] \to \mathbf{C}$ be the unique curve parametrized by arc-length satisfying

$$\gamma_\mu(0) = 0 \text{ and } \mathbf{t}_{\gamma_\mu}(s) = \begin{cases} e^{-is} & \text{if } s \in [0, -\mu]; \\ e^{i(s+2\mu)} & \text{if } s \in [-\mu, -\mu + \omega]; \\ e^{-i(s-2\omega)} & \text{if } s \in [-\mu + \omega, 2\omega - \theta_1]. \end{cases}$$

Notice that $\mathbf{t}_{\gamma_\mu}(0) = 1$, $\mathbf{t}_{\gamma_\mu}(2\omega - \theta_1) = z$ and γ_μ is a concatenation of three arcs of circles of radius 1. Moreover, $\min \theta_{\gamma_\mu} = \mu$ and $\max \theta_{\gamma_\mu} = \mu + \omega$, where θ_{γ_μ} is the argument of $\mathbf{t}_{\gamma_\mu}$ satisfying $\theta_{\gamma_\mu}(0) = 0$. Consequently, γ_μ has amplitude ω. Further,

$$\gamma_\mu(2\omega - \theta_1) = \left(\int_\mu^0 + \int_\mu^{\mu+\omega} + \int_{\theta_1}^{\mu+\omega} \right) e^{is}\, ds = (-i + iz) + 4 \sin\left(\frac{\omega}{2}\right) e^{i\left(\mu + \frac{\omega}{2}\right)}.$$

Thus, as μ increases from $\theta_1 - \omega$ to 0, the endpoint of γ_μ traverses an arc of $C_{r(\omega)}(-i + iz)$. Suppose that there exists $\eta \in \hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ of amplitude ω, and let $\eta \colon [0, L] \to \mathbf{C}$ be parametrized by arc-length. Let θ_η be the argument of η satisfying $\theta_\eta(0) = 0$, take $\mu = \min \theta_\eta$ and define

$$g \colon [0, L] \to \mathbf{R} \quad \text{by} \quad g(s) = \left\langle \eta(s) - \gamma_\mu(2\omega - \theta_1)\,,\, e^{i\left(\mu + \frac{\omega}{2}\right)} \right\rangle.$$

Then the same reasoning used to establish (4.16) and (6.3) shows that $g(L) \geq 0$. This implies that $\eta(L) = q$ lies on or to the left of the line through $\gamma_\mu(2\omega - \theta_1)$ having direction $\exp\left(i\left(\mu + \frac{\omega - \pi}{2}\right)\right)$. This line is tangent to $C_{r(\omega)}(-i + iz)$ at this point, therefore q cannot lie inside of this circle. This proves the assertion about $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$. Since the latter contains $\mathcal{L}_{-1}^{+1}(Q; \theta_1)$ as a subset, the proof is complete. □

7. The components of $\mathcal{L}_{\kappa_1}^{\kappa_2}(P, Q)$ for $\kappa_1 \kappa_2 < 0$

(7.1) Theorem D. *Let $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$ and $\theta_1 \in \mathbf{R}$ satisfy $e^{i\theta_1} = z$. Then the following assertions about $\mathcal{L}_{-1}^{+1}(Q; \theta_1)$ are equivalent:*

- (i) *$\mathcal{L}_{-1}^{+1}(Q; \theta_1)$ is disconnected.*
- (ii) *$|\theta_1| < \pi$ and q lies in the region depicted in fig. 2.*
- (iii) *$|\theta_1| < \pi$ and there exist condensed curves, but not critical curves, in $\mathcal{L}_{-1}^{+1}(Q)$.*
- (iv) *$|\theta_1| < \pi$ and there exist condensed curves in $\mathcal{L}_{-1}^{+1}(Q)$, but no condensed curve is homotopic to a diffuse curve within $\mathcal{L}_{-1}^{+1}(Q)$.*
- (v) *There exists $\gamma \in \mathcal{L}_{-1}^{+1}(Q; \theta_1)$ which does not lie in the same component as $B_{\gamma, 1, \frac{1}{2}}$.*

Furthermore, if $\mathcal{L}_{-1}^{+1}(Q; \theta_1)$ is disconnected, then it has exactly two components; one of them is $\mathcal{U}_c$ and the other is $\mathcal{U}_d \subset \mathcal{L}_{-1}^{+1}(Q; \theta_1)$, and both are homeomorphic to $\mathbf{E}$, hence contractible.[†]

Proof. We know from (4.3) and (5.19) that each of $\mathcal{U}_c$, $\mathcal{U}_d \subset \mathcal{L}_{-1}^{+1}(Q; \theta_1)$ is homeomorphic to $\mathbf{E}$, hence connected. By (5.10), $\mathcal{U}_d \neq \emptyset$. By (6.1), $\bar{\mathcal{U}}_c \cup \bar{\mathcal{U}}_d = \mathcal{L}_{-1}^{+1}(Q; \theta_1)$, and $\bar{\mathcal{U}}_c \cap \bar{\mathcal{U}}_d$ consists of all the critical curves in $\mathcal{L}_{-1}^{+1}(Q; \theta_1)$. Thus, the latter has at most two connected components. It has exactly two if and only if $\bar{\mathcal{U}}_c \neq \emptyset$ but $\bar{\mathcal{U}}_c \cap \bar{\mathcal{U}}_d = \emptyset$, that is, if and only if there exist condensed curves, but not critical curves. This proves the last assertion of the theorem and also the equivalence (i)⇔(iii). The equivalence (ii)⇔(iii) was proved in (6.5).

Suppose that $s \mapsto \gamma_s \in \mathcal{L}_{-1}^{+1}(Q; \theta_1)$ is a path joining a condensed curve to a diffuse curve. Let θ_s be the argument of $\mathbf{t}_{\gamma_s}$ satisfying $\theta_s(0) = 0$. By continuity, there must exist $s_0 \in (0, 1)$ such that

$$\max_{t \in [0,1]} \theta_{s_0}(t) - \min_{t \in [0,1]} \theta_{s_0}(t) = \pi,$$

that is, there must exist s_0 such that γ_{s_0} is critical. Hence, (iii)⇒(iv).

If (iv) holds and γ is condensed, then it cannot lie in the same component of $\mathcal{L}_{-1}^{+1}(Q; \theta_1)$ as $B_{\gamma, 1, \frac{1}{2}}$, for the latter curve is diffuse. Hence, (iv)⇒(v).

Finally, it is obvious that (v)⇒(i). □

[†]Recall that $\mathbf{E}$ denotes the separable Hilbert space and $B_{\gamma, 1, \frac{1}{2}}$ is obtained from γ by attaching a figure eight curve (at $t = 1/2$). See (5.4) and fig. 8(d).

(7.2) Corollary. *Let $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$ and $\theta_1 \in \mathbf{R}$ satisfy $e^{i\theta_1} = z$. If $|\theta_1| \geq \pi$, then $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ is connected. If $|\theta_1| > \pi$, then $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ is homeomorphic to $\mathbf{E}$, hence contractible.*

Proof. The first assertion is an immediate consequence of (7.1). If $|\theta_1| > \pi$ then $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ can only contain diffuse curves, and we know from (5.19) that $\mathcal{U}_d$ is homeomorphic to $\mathbf{E}$. □

Remark. The results of §5 go through to show that $\mathcal{L}^{+1}_{-1}(Q; \theta_1) = \mathcal{T} \cup \mathcal{U}_d$ is also contractible when $\theta_1 = \pm\pi$. Of course, if $|\theta_1| < \pi$ then $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ need not even be connected. We shall prove in the sequel [9] that it may also be contractible, or connected but not contractible, depending on Q.

(7.3) Corollary. *Let $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$ and $\theta_1 \in \mathbf{R}$ satisfy $e^{i\theta_1} = z$. Then the subset $\mathcal{L}^{+1}_{-1}(Q; \theta_1)$ is either a connected component or the union of two contractible components of $\mathcal{L}^{+1}_{-1}(Q)$. The latter can occur only if $|\theta_1| < \pi$, that is, for at most one value of θ_1.* □

(7.4) Theorem E. *Let $P = (p, w)$, $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$, $\kappa_1 < 0 < \kappa_2$ and let θ_1 satisfy $e^{i\theta_1} = z\bar{w}$.*

(a) *If $|\theta_1| \geq \pi$, then the subspace $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q; \theta_1)$ consisting of all curves having total turning θ_1 is a contractible connected component of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$, homeomorphic to $\mathbf{E}$.*

(b) *If $|\theta_1| < \pi$, then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q; \theta_1)$ has at most two components. It is disconnected if and only if*

$$\hat{q} = \tfrac{2}{\rho_2 - \rho_1}\bar{w}\big[(q - p) + \tfrac{i}{2}(\rho_1 + \rho_2)(z - w)\big] \qquad (\rho_i = \tfrac{1}{\kappa_i}, i = 1, 2)$$

lies in the region depicted in fig. 2 with $z\bar{w}$ in place of z. In this case, one component consists of all condensed and the other of all diffuse curves in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q; \theta_1)$, and both are homeomorphic to $\mathbf{E}$.

Proof. This is just a corollary of (3.3(a)), (7.1) and (7.2) □

We emphasize that the subspace of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ which contains curves having least total turning, described in (b), does not have to be contractible even if it is connected. Observe also that we may replace $\mathcal{L}$ by $\mathcal{C}$ invoking (1.16).

8. The homeomorphism class of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ for $\kappa_1\kappa_2 \geq 0$

An admissible plane curve γ will be called *locally convex* if either $\kappa_\gamma > 0$ a.e. or $\kappa_\gamma < 0$ a.e.. Notice that $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ consists of locally convex curves if and only if $\kappa_1\kappa_2 \geq 0$. This corresponds to parts (b)–(e) of (3.3). The topology of these spaces is very simple.

Suppose that $\gamma\colon [0, 1] \to \mathbf{C}$ is an admissible curve such that $\kappa_\gamma > 0$ a.e. and $\Phi_\gamma(0) = (0, 1)$. By (12), any argument $\theta\colon [0, 1] \to \mathbf{R}$ of $\mathbf{t}_\gamma$ must be strictly increasing; in particular, the total turning θ_1 of γ is positive. Thus, γ may be *parametrized by* its *argument* $\theta \in [0, \theta_1]$. By the chain rule,

$$\dot{\gamma}(\theta) = \rho(\theta)e^{i\theta} \quad (\theta \in [0, \theta_1]), \tag{36}$$

where $\rho\colon [0, \theta_1] \to (0, +\infty)$ is the radius of curvature of γ.[†]

(8.1) Theorem F. *Let P, $Q \in UT\mathbf{C}$ and suppose that either $\kappa_1 \geq 0$ or $\kappa_2 \leq 0$. Then $\mathcal{L}^{\kappa_2}_{\kappa_1}(P, Q)$ has infinitely many connected components, one for each realizable total turning. All of these components are homeomorphic to $\mathbf{E}$, hence contractible.*

Proof. Using an Euclidean motion if necessary, we may assume that $P = (0, 1)$. Further, by reversing the orientation of all curves, we pass from the case where $\kappa_2 \leq 0$ to the case where $\kappa_1 \geq 0$.

Let $Q = (q, z)$ and $e^{i\theta_1} = z$. The subspace $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q; \theta_1)$ is both open and closed in $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q)$ (but it may be empty, see (5.9)). In particular, two curves which have different total turnings cannot lie in the same component of $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q)$. For any $k \in \mathbf{N}$, we may concatenate a curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q)$ with a circle of curvature in (κ_1, κ_2) traversed k times. This shows that the number of components is infinite.

Suppose that $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q; \theta_1) \neq \emptyset$. Since $\kappa_1 \geq 0$ by hypothesis, we may reparametrize all curves in $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q; \theta_1)$ by the argument $\theta \in [0, \theta_1]$ of their unit tangent vectors using (1.18). Choose any $\gamma_0 \in \mathcal{L}^{\kappa_2}_{\kappa_1}(Q; \theta_1)$ and define a map H on $[0, 1] \times \mathcal{L}^{\kappa_2}_{\kappa_1}(Q; \theta_1)$ by:

$$H(s, \gamma) = \gamma_s, \quad \gamma_s(\theta) = (1 - s)\gamma_0(\theta) + s\gamma(\theta) \quad (s \in [0, 1],\ \theta \in [0, \theta_1]).$$

[†]The idea of parametrizing a locally convex curve by the argument of its unit tangent vector is not new. It appears in [6], where it is attributed to W. Pohl. We do not know whether it is older than that.

Then $\gamma_s(0) = 0$, $\gamma_s(\theta_1) = q$ and the unit tangent vector $\mathbf{t}_{\gamma_s}$ to γ_s satisfies

$$\mathbf{t}_{\gamma_s}(\theta) = e^{i\theta} \quad \text{for all } \gamma \in \mathcal{L}^{\kappa_2}_{\kappa_1}(Q;\theta_1),\ s \in [0,1] \text{ and } \theta \in [0,\theta_1].$$

Consequently, each γ_s has total turning θ_1, $\Phi_{\gamma_s}(0) = (0,1)$ and $\Phi_{\gamma_s}(\theta_1) = Q$. Let ρ_0, $\rho\colon [0,1] \to (0,+\infty)$ denote the radii of curvature of γ_0, γ, respectively. It follows from (36) that the radius of curvature ρ_s of γ_s is given by:

$$\rho_s = (1-s)\rho_0 + s\rho.$$

Therefore, the curvature $\kappa_s = \frac{1}{\rho_s}$ of γ_s takes values in (κ_1,κ_2) and H is a contraction of $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q;\theta_1)$. We conclude that the latter is a connected component of $\mathcal{L}^{\kappa_2}_{\kappa_1}(Q)$ and, using (1.12(b)), that it is homeomorphic to $\mathbf{E}$. □

Possible total turnings of a curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ when $\kappa_1\kappa_2 > 0$. Let T denote the set of all total turnings which are realized by some curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$. If $P = (p,w)$, $Q = (q,z)$, then obviously

$$T \subset \left\{\theta_1 + 2k\pi : k \in \mathbf{Z}\right\}, \quad \text{where } e^{i\theta_1} = z\bar{w}.$$

If $\kappa_1\kappa_2 < 0$, this inclusion is an equality by (5.9(b)). However, it must be proper when $\kappa_1\kappa_2 \geq 0$. If κ_1, κ_2 are both positive, for instance, then, by (12) and the second paragraph of the above proof, T must have the form $\left\{\mu + 2k\pi : k \in \mathbf{N}\right\}$, where $\mu \in \mathbf{R}$ ($e^{i\mu} = z\bar{w}$) is the minimal attainable total turning in this space. The task at hand is to determine μ in terms of all parameters involved. We begin by introducing locally convex curves of infinite curvature, taking the hint of (36).

(8.2) Definition. Let $\kappa_0 = \frac{1}{\rho_0} > 0$, $P = (p,w)$, $Q = (q,z) \in \mathbf{C}\times\mathbf{S}^1$. Let $\theta_0 \in \mathbf{R}$ and $\theta_1 > 0$ be such that $e^{i\theta_0} = w$ and $e^{i\theta_1} = z\bar{w}$. We define the set $\hat{\mathcal{L}}^{+\infty}_{\kappa_0}(P,Q;\theta_1)$ to consist of all $\gamma\colon [\theta_0,\theta_0+\theta_1] \to \mathbf{C}$ satisfying $\gamma(\theta_0+\theta_1) = q$ and

$$\gamma(\theta) = p + \int_{\theta_0}^{\theta} \rho(\tau)e^{i\tau}\,d\tau \quad (\theta \in [\theta_0,\theta_0+\theta_1])$$

for some L^2 function $\rho\colon [\theta_0,\theta_0+\theta_1] \to [0,\rho_0]$. We set $\mathbf{t}_\gamma(\theta) = e^{i\theta}$ for all $\theta \in [\theta_0,\theta_0+\theta_1]$ by definition. As the notation indicates, the function ρ is the radius of curvature of γ.

Remark. Observe that γ is not regular at the points where $\rho(\theta) = 0$ (i.e., at the points where the curvature is $+\infty$), even though its unit tangent is defined at all points. The geometric interpretation is that at these points $\mathbf{t}_\gamma$ rotates counterclockwise while γ remains stationary.

(8.3) Lemma. *Let $P_0 = (p,w)$, $Q_0 = (q,z)$. Suppose $0 < \kappa_1 < \kappa_2$ or $\kappa_1 < \kappa_2 < 0$, and let $\rho_i = \frac{1}{\kappa_i}$. Then there exists a bijection between $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$ and $\hat{\mathcal{L}}^{+\infty}_1(Q_2)$ or $\hat{\mathcal{L}}^{+\infty}_1(Q_4)$, respectively, where*

$$Q_2 = \left(\tfrac{\bar{w}}{\rho_1-\rho_2}\left[(q-p) + i\rho_2(z-w)\right],\ z\bar{w}\right);$$
$$Q_4 = \left(\tfrac{\bar{z}}{\rho_1-\rho_2}\left[(q-p) + i\rho_1(z-w)\right],\ w\bar{z}\right);$$

which preserves the total turning of curves in the first case and reverses its sign in the second.

Proof. The proof is identical to that of items (b) and (d) of (3.3). □

It is slightly simpler to determine the minimal total turning $\hat{\tau}(P,Q)$ among curves in $\hat{\mathcal{L}}^{+\infty}_1(P,Q)$ than in $\mathcal{L}^{+\infty}_1(P,Q)$. We shall indicate how the answer changes if one works with $\mathcal{L}^{+\infty}_1(P,Q)$ afterwards.

(8.4) Definition. Let $r > 0$. An *r-circle* is a circle of radius r, oriented counterclockwise. In other words, an r-circle is a circle of curvature $\frac{1}{r}$. Let Γ be an r-circle and $P = (p,w) \in \mathbf{C}\times\mathbf{S}^1$. We shall write $P \in \Gamma$ if p lies on Γ and w is the unit tangent vector to Γ at p. For $A, B \in \Gamma$, we denote the closed (resp. open) arc of Γ going from A to B by $[A,B]$ (resp. (A,B)). Also, let $\arg(v)$ denote the unique $\theta \in [0,2\pi)$ such that $e^{i\theta} = \frac{v}{|v|}$ ($v \neq 0$).

The value of $\hat{\tau}(P,Q)$ may be found using the following elementary geometric construction.

(8.5) Construction. Let $P = (p,w)$, $Q = (q,z) \in \mathbf{C}\times\mathbf{S}^1$ (compare fig. 12).

1. Draw the 1-circles Γ and Γ' centered at $O = p + iw$ and $O' = q + iz$, respectively.

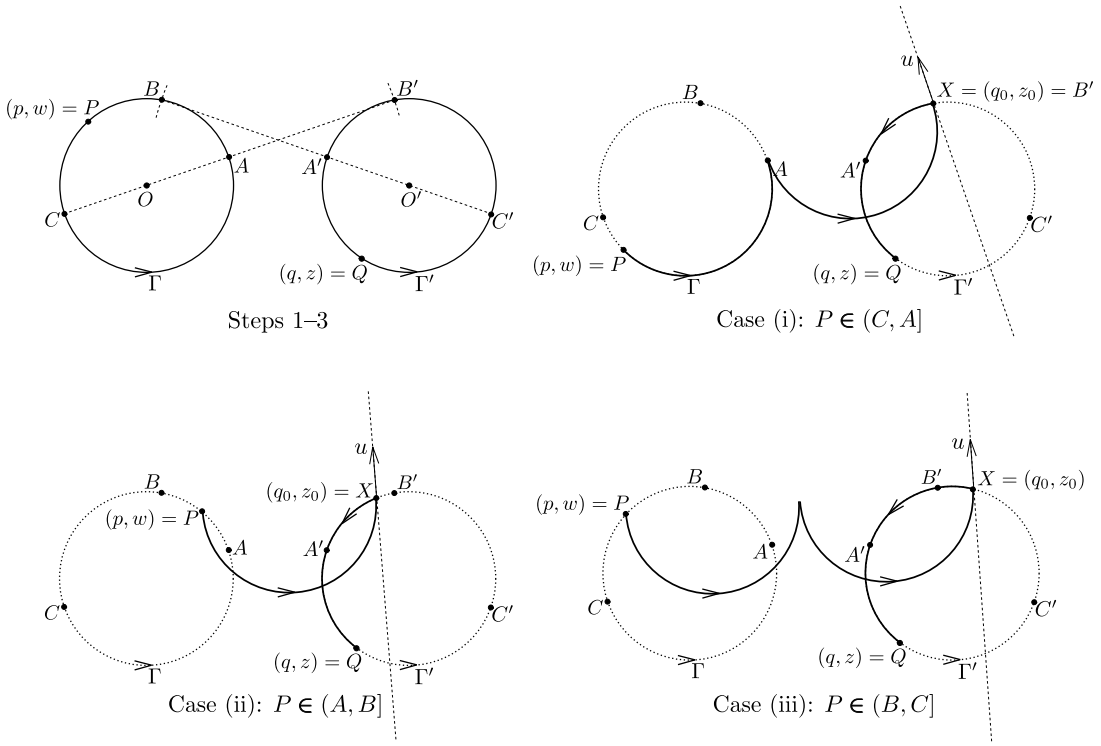


FIGURE 12. An illustration of (8.5) and (8.6). Depicted in bold are paths of minimal total turning in $\hat{\mathcal{L}}_1^{+\infty}(P,Q)$ for P satisfying (i), (ii) and (iii) of (8.5), and a fixed Q. In this example $n = 1$. The unit vector u is not drawn to scale.

2. Determine the smallest $n \in \{0, 1, \dots\}$ such the $(2n+1)$-circle Γ_{2n+1} centered at O intersects Γ'. Let Γ'_{2n+1} be the corresponding circle centered at O'.
3. Let $B \in \Gamma$ be the point of $\Gamma \cap \Gamma'_{2n+1}$ which lies on or to the left of $\vec{OO'}$, and let B' be the analogous point on Γ'. Let $A, C \in \Gamma$ be the points in $\Gamma \cap OB'$, with A between O and B'. Let $A', C' \in \Gamma'$ be the analogous points on Γ'.
4. Let $X = (q_0, z_0) \in \Gamma'$ and $u \in \mathbf{S}^1$ be defined as follows (cf. fig. 12 (i), (ii), (iii), where $n = 1$):
   (i) If $P \in (C, A]$, then $X = B'$ and u is the unit tangent to Γ_{2n+1} at q_0.
   (ii) If $P \in (A, B]$, then q_0 is the intersection of the $2n$-circle $\Gamma_{2n}(P)$ centered at P with $(B', A']$; $u, z_0 \in \mathbf{S}^1$ satisfy $(q_0, z_0) \in \Gamma'$ and $(q_0, u) \in \Gamma_{2n}(P)$.
   (iii) If $P \in (B, C]$, then q_0 is the intersection of the $(2n+2)$-circle $\Gamma_{2n+2}(P)$ centered at P with $(C', B']$; $u, z_0 \in \mathbf{S}^1$ satisfy $(q_0, z_0) \in \Gamma'$ and $(q_0, u) \in \Gamma_{2n+2}(P)$.

(8.6) Proposition. *Let the notation be as in (8.5). Then the minimal total turning among curves in $\hat{\mathcal{L}}_1^{+\infty}(P,Q)$ is given by:*

$$\hat{\tau}(P,Q) = \arg(u\bar{w}) + \arg(z_0\bar{u}) + \arg(z\bar{z}_0) + \begin{cases} 2n\pi & \text{in cases (i) and (iii);} \\ 2(n-1)\pi & \text{in case (ii).} \end{cases}$$

Remark. Note that if Γ and Γ' intersect (that is, if $n = 0$) or if Γ_{2n+1} is tangent to Γ', then $A = B$ and $A' = B'$. Hence case (ii) cannot arise, and

$$\hat{\tau}(P,Q) = \arg(u\bar{w}) + \arg(z_0\bar{u}) + \arg(z\bar{z}_0) + 2n\pi.$$

Proof of (8.6). To abbreviate the notation, let us suppose that $P = (p, w) = (0, 1) \in \mathbf{C}\times\mathbf{S}^1$. Assume first that we are in case (i) or (iii), that is, $P \notin (A, B]$. Set $\varphi_0 = \arg(u\bar{w}) + 2n\pi = \arg(u) + 2n\pi$. Define $\rho_0 \colon [0, \varphi_0] \to \mathbf{R}$ by

$$\rho_0(\theta) = \begin{cases} 1 & \text{if } \langle e^{i\theta},\, iu\rangle \le 0 \\ 0 & \text{if } \langle e^{i\theta},\, iu\rangle > 0 \end{cases}$$

and let $\gamma\colon [0,\varphi_0] \to \mathbf{C}$ be the unique curve in $\hat{\mathcal{L}}_1^{+\infty}(Q)$ parametrized by argument whose radius of curvature is ρ_0; by construction, $\Phi_\gamma(\varphi_0) = (q_0, u)$. ($\gamma$ is the part of the curve in bold in figs. 12(i) and 12(iii) which joins p to q_0)

Let $\eta \in \hat{\mathcal{L}}_1^{+\infty}(Q_1)$ for some $Q_1 \in UT\mathbf{C}$, $\eta\colon [0,\varphi_1] \to \mathbf{C}$ parametrized by argument. Let $\rho\colon [0,\varphi_1] \to [0,1]$ denote its radius of curvature. Also, let H be the half-plane

$$H = \big\{a \in \mathbf{C} : \langle a - q_0, iu\rangle \geq 0\big\}.$$

(H is the closed region to the left of the dashed line in figs. 12(i) and 12(iii).) Define $f\colon [0,\varphi_0] \to \mathbf{R}$ and $g\colon [0,\varphi_1] \to \mathbf{R}$ by

$$\begin{matrix} f(\theta) = \langle \gamma(\theta) - q_0,\, iu\rangle \\ g(\theta) = \langle \eta(\theta) - q_0,\, iu\rangle \end{matrix}, \quad \text{so that} \quad \begin{matrix} f'(\theta) = \rho_0(\theta)\left\langle e^{i\theta},\, iu\right\rangle \\ g'(\theta) = \rho(\theta)\left\langle e^{i\theta},\, iu\right\rangle \end{matrix}$$

Then $f(0) = g(0) = -\langle q_0,\, iu\rangle \geq 0$ and $g'(\theta) \geq f'(\theta)$ for all $\theta \in [0,\varphi_0] \cap [0,\varphi_1]$. Hence $g(\theta) \geq f(\theta) \geq 0$ for all such θ. Moreover,

$$g'(\theta) = \rho(\theta)\left\langle e^{i\theta}, iu\right\rangle = \rho(\theta)\left\langle e^{i\theta}, e^{i(\varphi_0 + \frac{\pi}{2})}\right\rangle \geq 0 \quad \text{for } \theta \in [\varphi_0, \varphi_0 + \pi] \cap [0,\varphi_1].$$

We conclude that $g(\theta) \geq 0$ for all $\theta \in [0,\varphi_0 + \pi] \cap [0,\varphi_1]$. Consequently, if $\eta(\varphi_1) \notin H$, then

$$\varphi_1 > \varphi_0 + \pi = \arg(u\bar{w}) + (2n+1)\pi. \tag{37}$$

If we follow the path γ, rotate (counterclockwise and without moving) by $\arg(z_0\bar{u})$ and then traverse Γ' until we reach Q, we obtain a curve $\gamma_0 \in \hat{\mathcal{L}}_1^{+\infty}(Q)$ (depicted in bold in figs. 12(i) and 12(iii)) whose total turning is exactly

$$\arg(u\bar{w}) + \arg(z_0\bar{u}) + \arg(z\bar{z}_0) + 2n\pi.$$

Suppose that the total turning of $\zeta \in \hat{\mathcal{L}}_1^{+\infty}(Q)$ is smaller than this, i.e., at most equal to

$$\arg(u\bar{w}) + \arg(z_0\bar{u}) + \arg(z\bar{z}_0) + (2n-2)\pi.$$

Choose $Q_1 = (q_1, z_1) \in \Gamma$ on the arc (Q, X), with $q_1 \notin H$; this is possible because points sufficiently close to, but lying before X in the orientation of Γ', all lie outside of H. Concatenating ζ with the arc $[Q, Q_1]$, we obtain a curve $\eta \in \hat{\mathcal{L}}_1^{+\infty}(Q_1)$ whose total turning is

$$\begin{aligned} \varphi_1 &= \arg(u\bar{w}) + (2n-2)\pi + \arg(z_0\bar{u}) + \big(\arg(z\bar{z}_0) + \arg(z_1\bar{z})\big) \\ &< \arg(u\bar{w}) + (2n-2)\pi + \pi + 2\pi = \arg(u\bar{w}) + (2n+1)\pi. \end{aligned}$$

Since $q_1 = \eta(\varphi_1) \notin H$, this contradicts (37). Hence,

$$\tau(P,Q) = \arg(u\bar{w}) + \arg(z_0\bar{u}) + \arg(z\bar{z}_0) + 2n\pi$$

in cases (i) and (iii), as claimed. The proof in case (ii) goes through mutatis mutandis; the main difference is that we set $\varphi_0 = \arg(u\bar{w}) + (2n-2)\pi$ at the beginning. □

(8.7) Corollary. *Let the notation be as in (8.5). Then the minimal total turning $\tau(P,Q)$ among curves in $\mathcal{L}_1^{+\infty}(P,Q)$ is given by:*

$$\tau(P,Q) = \arg(u\bar{w}) + \arg(z_0\bar{u}) + \arg(z\bar{z}_0) + \begin{cases} 2n\pi & \text{in cases (i) and (iii);} \\ 2(n-1)\pi & \text{in case (ii);} \end{cases}$$

unless $z_0 = -u$ or $z = z_0$, in which case 2π should be added to the expression on the right.

Proof. We shall only provide an outline of the proof. To say that $z_0 = -u$ is the same as to say that the curve γ constructed in the proof of (8.6) is tangent to Γ'. If this occurs or if $z = z_0$, then γ_0 must be an isolated point of $\hat{\mathcal{L}}_1^{+\infty}(P,Q)$. Hence, there does not exist a curve in $\mathcal{L}_1^{+\infty}(P,Q)$ having the same total turning $\hat{\tau}(P,Q)$ as γ_0.

If $z_0 \neq -u$ then for $\theta < \varphi_0$ very close to φ_0, $\gamma_0(\theta)$ lies inside the disk bounded by Γ'. If in addition $z \neq z_0$, then a slight modification of γ_0 yields a curve in $\mathcal{L}_1^{+\infty}(P,Q)$ of total turning $\hat{\tau}(P,Q)$. □

To determine the possible total turnings of curves in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$ for arbitrary κ_1,κ_2 satisfying $\kappa_1\kappa_2>0$, we first invoke (3.3(b)) or (3.3(d)) to find $Q\in\mathbf{C}\times\mathbf{S}^1$ with

$$\mathcal{L}^{\kappa_2}_{\kappa_1}(P_0,Q_0)\approx\mathcal{L}^{+\infty}_1(Q),$$

and then apply the previous corollary with $P=(0,1)\in\mathbf{C}\times\mathbf{S}^1$. As seen in (3.3), the homeomorphism above preserves the total turning of a curve in case $0<\kappa_1<\kappa_2$ and reverses its sign when $\kappa_1<\kappa_2<0$. We conclude that the total turning of a curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$ must have the form

$$\begin{array}{rll}\tau(P,Q)+2k\pi & \text{if } 0<\kappa_1<\kappa_2 & \text{or}\\ -(\tau(P,Q)+2k\pi) & \text{if } \kappa_1<\kappa_2<0 & \end{array}\qquad (k\in\mathbf{N},\ k\geq 0) \tag{38}$$

and every such number is realized as the total turning of some curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$.

Naturally, there is an analogous result for the minimal total turning of curves in $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$, involving (8.3) and (8.6) instead. In fact, taking note of the steps in the proofs of these results, it would be possible to write down a procedure for obtaining an explicit curve in $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$ having a required (and realizable) total turning. We shall leave this task to the interested reader.

Possible total turnings of a curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ when $\kappa_1\kappa_2=0$.

(8.8) Proposition. *Let $Q=(q,z)\in\mathbf{C}\times\mathbf{S}^1$ and let $\tau_0(Q)$ denote the smallest total turning among curves in $\mathcal{L}^{+\infty}_0(Q)$. Let $S=\{p\in\mathbf{C}:p\neq 0$ and $0<\arg(p)<\arg(z)\}$. Then*

$$\tau_0(Q)=\begin{cases}\arg(z) & \text{if } q\in S \text{ or } \arg(z)>\pi;\\ \arg(z)+2\pi & \text{otherwise.}\end{cases}$$

Proof. Let $v\in\mathbf{S}^1$, $\pi<\arg(v)<2\pi$. Write $q=(a_1-a_2)+bi+cv$, where $a_1,a_2,b,c\geq 0$. Consider the following curve: Starting at 0, we follow a straight line segment of length a_1 along the x-axis, then we increase the argument from 0 to $\frac{\pi}{2}$ without moving, follow a line segment of length b, increase the argument from $\frac{\pi}{2}$ to π, follow a line segment of length a_2, increase the argument from π to $\arg(v)$, follow a straight line segment of length c to reach q, and finally increase the argument from $\arg(v)$ to $\arg(z)+2\pi$. Even though this curve does not lie in $\mathcal{L}^{+\infty}_0(Q)$, it is easy to see that it can be modified to obtain a curve in the latter space which has total turning $\arg(z)+2\pi$. Hence, we always have

$$\arg(z)\leq\tau_0(Q)\leq\arg(z)+2\pi.$$

Moreover, if $\arg(z)>\pi$, then we can choose v so that $\pi<\arg(v)<\arg(z)$ and, in the final step described above, increase the argument from $\arg(v)$ to $\arg(z)$ instead. Thus,

$$\tau_0(Q)=\arg(z)\quad\text{if } \arg(z)>\pi.$$

Similarly, if $q\in S$ and $\arg(z)\leq\pi$, then we can write $q=a+bv$ for $a,b\geq 0$ and $v\in\mathbf{S}^1$ with $0<\arg(v)<\arg(z)$, and proceed as above to conclude that $\tau_0(Q)=\arg(z)$.

Conversely, assume that $\gamma\in\mathcal{L}^{+\infty}_0(Q)$ has total turning $\arg(z)$, for $\arg(z)\leq\pi$. Let γ be parametrized by argument, and let $\rho\colon[0,\arg(z)]\to(0,+\infty)$ denote its radius of curvature. Then

$$q=\int_0^{\arg(z)}\rho(\theta)e^{i\theta}d\theta$$

must belong to S, since S is a convex cone. □

More generally, let $P_0,Q_0\in UT\mathbf{C}$. Still assuming that one of $\kappa_1<\kappa_2$ is zero, we can, with the help of (3.3(c)) and (3.3(e)), find $Q\in UT\mathbf{C}$ and a homeomorphism

$$\mathcal{L}^{\kappa_2}_{\kappa_1}(P_0,Q_0)\approx\mathcal{L}^{+\infty}_0(Q)$$

which preserves the total turning in case $\kappa_1=0$ and reverses its sign when $\kappa_2=0$. Hence, the set T_0 of possible total turnings of a curve in $\mathcal{L}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$ is given by:

$$\begin{aligned}T_0&=\big\{\tau_0(Q)+2k\pi:k\in\mathbf{N}\big\}\qquad\text{if } \kappa_1=0 \text{ and}\\ T_0&=\{-\big(\tau_0(Q)+2k\pi\big):k\in\mathbf{N}\}\quad\text{if } \kappa_2=0,\end{aligned} \tag{39}$$

where the value of $\tau_0(Q)$ was found in (8.8).

9. The components of spaces of curves on complete flat surfaces

By a *flat surface* we mean a connected Riemannian 2-manifold whose Gaussian curvature is identically zero. It is permitted that a flat surface S be nonorientable. Even when this is the case, we can still speak of the unsigned curvature of a curve on S: If $\gamma\colon [0,1] \to S$ is of class C^2 and $\mathbf{t}\colon [0,1] \to UTS$ is its unit tangent vector, we define its *curvature* $\kappa_\gamma\colon [0,1] \to [0,+\infty)$ by

$$\kappa_\gamma = \frac{1}{|\dot\gamma|}\left|\left\langle \frac{D\mathbf{t}}{dt}, \mathbf{n}\right\rangle\right|$$

where now $\mathbf{n}(t)$ denotes any of the two unit vectors in $TS_{\gamma(t)}$ orthogonal to $\mathbf{t}(t)$. Given $u, v \in UTS$ and $\kappa_0 > 0$, one can then define $\mathcal{C}^{+\kappa_0}_{-\kappa_0}(u,v)$ by replacing condition (ii) in (1.1) by:

(ii′) $\kappa_\gamma(t) < \kappa_0$ for each $t \in [0,1]$.

Another approach would be to use the orientable double cover $\tilde S$ of S to define $\mathcal{C}^{+\kappa_0}_{-\kappa_0}(u,v)$ (and $\mathcal{L}^{+\kappa_0}_{-\kappa_0}(u,v)$) as two copies of a space of this form over $\tilde S$.

Remark. A complete flat surface must be homeomorphic to one of the following five: $\mathbf{C}$ itself, a cylinder $\mathbf{S}^1 \times \mathbf{R}$, an open Möbius band, a torus or a Klein bottle. This is essentially a corollary of the following result, cf. [5], p. 319.

(9.1) Theorem (Killing-Hopf). *Any complete flat surface is isometric to the quotient of the Euclidean plane $\mathbf{C}$ by a group of isometries acting freely and properly discontinuously on $\mathbf{C}$.* □

Hence, if S is a complete flat surface, there exists a covering map $\mathbf{C} \to S$ which is also a local isometry. Any curve on S may thus be lifted to a plane curve whose curvature is the same as that of the original curve, with the proviso that we ignore its sign if S is nonorientable.

Let $\mathrm{pr}\colon UT\mathbf{C} \to UTS$ denote the natural projection induced by the covering map. To avoid confusion, let us reserve the notation $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$ for spaces of plane curves and denote the corresponding space of curves on S by $\mathcal{C}S^{\kappa_2}_{\kappa_1}(u,v)$, with the convention that $\kappa_1 = -\kappa_2 < 0$ if S is nonorientable.

(9.2) Proposition. *Let S be a complete flat surface, $u, v \in UTS$, $-\infty \le \kappa_1 < \kappa_2 \le +\infty$ and $P \in UT\mathbf{C}$ be a fixed element of $\mathrm{pr}^{-1}(u)$. Then $\mathcal{C}S^{\kappa_2}_{\kappa_1}(u,v)$ is homeomorphic to $\sum_{Q\in\mathrm{pr}^{-1}(v)} \mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$, where the homeomorphism maps a curve in the latter to its image under the quotient map $\mathbf{C} \to S$.* □

Here $\sum$ denotes topological sum. Clearly, this decomposition is sufficient to determine the connected components of $\mathcal{C}S^{\kappa_2}_{\kappa_1}(u,v)$ explicitly, using (1.16) and (7.4) if $\kappa_1\kappa_2 < 0$ or (8.1) if $\kappa_1\kappa_2 \ge 0$.

(9.3) Corollary. *Let S be a complete flat surface, $\kappa_1 < \kappa_2$ and $u, v \in UTS$. Then $\mathcal{C}S^{\kappa_2}_{\kappa_1}(u,v)$ is nonempty and has an infinite number of connected components.*

Proof. By (5.9) and the remark which follows it, $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$ is always nonempty. The assertion is thus an immediate consequence of (9.2). □

Notice that S may even be compact (e.g., the flat torus or the flat Klein bottle). This should be compared to the case of $S = \mathbf{S}^2$, where, at least when $u = v$, the number of components of $\mathcal{C}S^{\kappa_2}_{\kappa_1}(u,v)$ is finite for any choice of $\kappa_1 < \kappa_2$ (see §7 of [10]). This is actually not so surprising, since the fundamental group of $UT\mathbf{C}$ is isomorphic to $\mathbf{Z}$, but that of $UT\mathbf{S}^2 \approx \mathbf{SO}_3$ is isomorphic to $\mathbf{Z}_2$. See also (1.8), which shows that $\mathcal{C}S^{\kappa_2}_{\kappa_1}(u,v)$ may be empty for more general surfaces.

(9.4) Corollary. *Let S be a complete flat surface and $u, v \in UTS$. Let $\eta \in \mathcal{C}S^{+\infty}_{-\infty}(u,v)$ and suppose that $\kappa_1 < \kappa_2$ satisfy $\kappa_1\kappa_2 < 0$. Then there exists $\gamma \in \mathcal{C}S^{\kappa_2}_{\kappa_1}(u,v)$ lying in the same connected component of $\mathcal{C}S^{+\infty}_{-\infty}(u,v)$ as η.*

Proof. Take $P \in UT\mathbf{C}$ such that $\mathrm{pr}(P) = u$. Let $\tilde\eta$ be the lift of η to $\mathbf{C}$ with initial frame P; let Q be its final frame and θ_1 its total turning. By (5.9(b)), $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q;\theta_1)$ is nonempty. Let $\tilde\gamma$ be one of its elements. Then the projection γ of $\tilde\gamma$ on S satisfies the conclusion of the corollary because of (9.2) and the fact that $\tilde\eta$, $\tilde\gamma$ lie in the same component of $\mathcal{C}^{+\infty}_{-\infty}(P,Q)$. □

Again, the analogue of this result does not hold for a general surface S, e.g., for $S = \mathbf{H}^2$; see (1.8). It is also false for a flat surface if $\kappa_1\kappa_2 \ge 0$. To see this, let $P, Q \in UT\mathbf{C}$ satisfy $\mathrm{pr}(P) = u$, $\mathrm{pr}(Q) = v$, choose $\tilde\eta \in \mathcal{C}^{+\infty}_{-\infty}(P,Q)$ to have a total turning which is unattainable for curves in $\mathcal{C}^{\kappa_2}_{\kappa_1}(P,Q)$ and let η be the projection of $\tilde\eta$ on S.

10. Final remarks

Spaces of curves with curvature in a closed interval. In both [2] and [3], Dubins worked with the set $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(Q)$ of definition (2.3) (but with the C^1 topology), where the curvatures are restricted to lie in a *closed* interval. This choice is motivated by the fact that these spaces, unlike those of the form $\mathcal{L}^{+\kappa_0}_{-\kappa_0}(Q)$, always contain curves of minimal length (see prop. 1 in [2]). All of the main results in our paper concerning the topology of $\mathcal{L}^{\kappa_2}_{\kappa_1}(P,Q)$ have analogues for $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$. We shall now briefly indicate the modifications which are necessary.

Notice that $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$ is *not* a Banach manifold, and that the analogue of (1.6) is false for these spaces, as shown by the example in (1.3). In contrast, (1.17) and (1.18) still hold when $\mathcal{M} = \hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$. The important corollary (3.5) has the following analogue, whose proof differs from that of (3.3) only by obvious modifications.

(10.1) Proposition. *Let P, $Q \in UT\mathbf{C}$ and $\kappa_1 < \kappa_2$ be finite. Then there exists a homeomorphism between $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$ and a space of type $\hat{\mathcal{L}}^{+1}_{-1}(Q_0)$, $\hat{\mathcal{L}}^{1}_{0}(Q_0)$ or $\hat{\mathcal{L}}^{2}_{1}(Q_0)$, according as $\kappa_1\kappa_2 < 0$, $\kappa_1\kappa_2 = 0$ or $\kappa_1\kappa_2 > 0$, respectively. Moreover, this homeomorphism preserves the total turning of curves unless $\kappa_1 < \kappa_2 \leq 0$, in which case it reverses the sign.* □

In case $\kappa_1\kappa_2 < 0$, we actually have $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q) \approx \hat{\mathcal{L}}^{+1}_{-1}(Q_1)$ with Q_1 as in the statement of (3.3). We leave the task of determining Q_0 in the other two cases to the interested reader. If one is willing to work with curves of infinite curvature, then (8.3) can also be used.

Let us denote by $\hat{\mathcal{U}}_c$, $\hat{\mathcal{U}}_d$ and $\hat{\mathcal{T}} \subset \hat{\mathcal{L}}^{+1}_{-1}(Q;\theta_1)$ the subspaces consisting of all condensed, diffuse and critical curves, where $Q = (q,z)$, $e^{i\theta_1} = z$ and $\hat{\mathcal{L}}^{+1}_{-1}(Q;\theta_1)$ consists of those curves in $\hat{\mathcal{L}}^{+1}_{-1}(Q)$ which have total turning θ_1. The analogue of (4.3), stating that $\hat{\mathcal{U}}_c$ is either empty or contractible, is, naturally, prop. (4.4), which was used to prove it. The results and proofs in §5 all need minimal or no modifications. In particular, $\hat{\mathcal{U}}_d$ is always nonempty and contractible.

The proof that $\partial\hat{\mathcal{U}}_d = \hat{\mathcal{T}}$ is the same as the one given in (6.1). The proof that $\partial\hat{\mathcal{U}}_c = \hat{\mathcal{T}}$, however, needs to be modified, since we have relied on (1.6). The idea is again to apply construction (4.7), but to all of γ, not just to some of its arcs as in the proof of (6.1). If γ is a critical curve, then the corresponding function f (see fig. 5) will attain the values $\pm\infty$, and at these points we need to assign weights, corresponding to the lengths of the line segments where θ_γ attains its maximum and minimum. Then we redefine $A(\sigma_-,\sigma_+)$ as the sum of the area under the graph of $f^{(\sigma_-,\sigma_+)}$ plus the weight at $+\infty$ minus the weight at $-\infty$. The process described in (4.7) will transform f into a bounded function of the same area, that is, it will decrease the amplitude of γ, making it a condensed curve.

The proofs of (4.16), (6.3) and (6.4), which deal with the existence of condensed and critical curves, go through unchanged; the only difference in the conclusions is that the corresponding regions $R_{\hat{\mathcal{U}}_c}$, R_σ and $R_{\hat{\mathcal{T}}}$ of the plane are now closed, instead of open. Thus, in the analogue of thm. (7.1), the region of fig. 1 should contain the two circles of radius 2, but not the circle of radius 4, and we cannot assert that $\hat{\mathcal{U}}_c$ and $\hat{\mathcal{U}}_d$ are homeomorphic to $\mathbf{E}$, only that they are contractible. The rest of the statement and the proof hold without modifications.

Similarly, the version of (8.1) for $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$ states that this space has one contractible connected component for each realizable total turning when $\kappa_1\kappa_2 \geq 0$. The proof is the same as that of (8.1) if $\kappa_1\kappa_2 > 0$. If $\kappa_1 = 0$, then we cannot really parametrize $\gamma \in \hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$ by argument. Nevertheless, the proof still works if we replace $\rho(\theta)\,d\theta$ by a measure $\mu(\theta)$ on the Borel subsets of $[0,\theta_1]$ which has an atom at θ if the curvature of γ vanishes at $\gamma(\theta)$; note that the convex combination of two measures is again a measure. The case where $\kappa_2 = 0$ can be deduced from this one by reversing orientations.

Let $P_0 = (p,w)$, $Q_0 = (q,z) \in \mathbf{C} \times \mathbf{S}^1$. The determination of the set of realizable total turnings in $\hat{\mathcal{L}}^{+\infty}_{1}(P,Q)$ was presented in (8.6), and the possible total turnings of curves in $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$ for $\kappa_1\kappa_2 > 0$ are given by (38) if we replace $\tau(P,Q)$ by $\hat{\tau}(P,Q)$, where $P = (0,1)$ and $Q = Q_2$ or Q_4, as in (8.3). Similarly, let $Q = Q_3$ or Q_5 be as in (3.3), depending on whether κ_1, κ_2 are both non-negative or both non-positive. Then the set T_0 of possible total turnings of curves in $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P_0,Q_0)$

for $\kappa_1\kappa_2 = 0$ is given by (39) if we replace $\tau_0(Q)$ by $\hat{\tau}_0(Q)$, where, in the notation of (8.8),

$$\hat{\tau}_0(Q) = \begin{cases} \arg(z) & \text{if } q \in \overline{S} \text{ or } \arg(z) > \pi; \\ \arg(z) + 2\pi & \text{otherwise;} \end{cases} \quad \text{for } Q = (q, z).$$

A few conjectures of Dubins. All of the results in the next proposition were conjectured by Dubins in §6 of [3].

(10.2) Proposition. *Let $q \in \mathbf{C}$, $\theta_1 \in \mathbf{R}$, $z = e^{i\theta_1}$ and $Q = (q, z)$. Then:*

(a) *The set of all $(q, \theta_1) \in \mathbf{C} \times \mathbf{R}$ such that $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ is disconnected is a bounded subset of $\mathbf{C} \times \mathbf{R}$, neither open nor closed.*[†]

(b) *$\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ has at most two components.*

(c) *If $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ is disconnected, then one component ($\hat{\mathcal{U}}_d$) contains curves of arbitrarily large length, while the supremum of the lengths of curves in the other component ($\hat{\mathcal{U}}_c$) is finite.*

(d) *Every point of $\mathbf{C}$ lies in the image of some $\gamma \in \hat{\mathcal{U}}_d$, while the images of curves in $\hat{\mathcal{U}}_c$ are contained in a bounded subset of $\mathbf{C}$.*

Proof. Parts (a) and (b) are immediate from the analogue of (7.1) for $\hat{\mathcal{L}}$. As discussed above, $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ is disconnected if and only if $|\theta_1| < \pi$ and q lies in the region in fig. 2 including the circles of radius 2 but not the circle of radius 4. Suppose that q does lie in this region. Choose $\hat{\omega} \in (\theta_1, \pi)$ such that

$$|q - \operatorname{sign}(\theta_1) i (z - 1)| < 4 \sin\Big(\frac{\hat{\omega}}{2}\Big).$$

Then (6.6) implies that there does not exist any curve in $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ having amplitude in $[\hat{\omega}, \pi]$. The assertions about $\hat{\mathcal{U}}_c$ in (c) and (d) now follow from (4.14). The assertions about $\hat{\mathcal{U}}_d$ are obvious, because, by (the version for $\hat{\mathcal{L}}$ of) (5.18), this subspace always contains curves of amplitude $\geq 2\pi$, and onto such a curve we may graft line segments of any direction and arbitrary length. □

Remark. Suppose that $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ is connected. Since critical curves can always be grafted and $\partial \hat{\mathcal{U}}_c = \mathcal{T}$, we deduce that there exist condensed curves having arbitrarily large length. Combining this with part (c) above, the following additional criterion is obtained: *$\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ is connected if and only if there exist condensed curves of arbitrarily large length in this space.*

As expected, there is a version of the foregoing proposition for $\mathcal{L}_{-1}^{+1}(Q; \theta_1)$. The corresponding assertions in (a) and (b) are immediate from (7.1), and the assertions about $\mathcal{U}_d$ are again obvious. As sets, $\mathcal{U}_c \subset \hat{\mathcal{U}}_c$, so the properties stated in (c) and (d) for $\mathcal{U}_c$ follow from those for $\hat{\mathcal{U}}_c$ unless q lies on the circle of radius 4 in fig. 2. In this case, $\mathcal{L}_{-1}^{+1}(Q; \theta_1)$ is disconnected, but $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ is not. One can prove directly that the length of any $\gamma \in \mathcal{U}_c$ must be smaller than that of the "canonical" critical curves of type $+-$ or $-+$ that were constructed in the proof of (6.3).

Conjectures on minimal length. Let $L(\gamma)$ denote the length of γ and suppose that $\hat{\mathcal{L}}_{-1}^{+1}(Q; \theta_1)$ is disconnected. We believe that the results developed here may be used to prove that if $m = \sup_{\gamma \in \hat{\mathcal{U}}_c} L(\gamma)$ and $M = \inf_{\gamma \in \hat{\mathcal{U}}_d} L(\gamma)$, then $m < M$; this is another conjecture of Dubins. It would be interesting, and probably useful for applications, to find the values corresponding to m and M for the more general spaces $\hat{\mathcal{L}}_{\kappa_1}^{\kappa_2}(Q)$.

We observed in (3.4) that the homeomorphisms of (3.3) or (10.1) do not preserve inequalities between lengths. Since they do map circles to circles and lines to lines, it could still be expected that the image of a curve which minimizes length under these homeomorphisms is likewise of minimal length. Unfortunately, this is false. Suppose for instance that we apply the homeomorphism $\hat{\mathcal{L}}_{-1}^{+1}(Q) \to \hat{\mathcal{L}}_{-1}^{100}(Q_0)$ to the Dubins path in fig. 4(b). It should be clear that its image, which again consists of a line segment and two arcs of circles of opposite orientation with the same amplitude as before, does not minimize length in $\hat{\mathcal{L}}_{-1}^{100}(Q_0)$, since in the latter space it is generally much more efficient to curve to the left than to the right, even if this yields a path of greater total turning.

[†]Actually, Dubins had guessed that this set would be bounded and open in $\mathbf{C} \times \mathbf{R}$.

In spite of this difficulty, we conjecture that Dubins' theorem that any shortest path in $\hat{\mathcal{L}}^{+\kappa_0}_{-\kappa_0}(P,Q)$ must be a concatenation of three pieces, each of which is either an arc of circle or a line segment, still holds for the spaces $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(P,Q)$, $\kappa_1\kappa_2 < 0$. For $\kappa_1\kappa_2 > 0$, we conjecture that a curve of minimal length is a concatenation of n arcs of circles of curvature κ_1 and κ_2. However, for fixed $P, Q \in UT\mathbf{C}$, we must have $\lim_{\kappa_1\to+\infty} n = \infty$. Indeed, a curve of this type has total turning at most $2n\pi$, and it follows from (8.3) and (8.6) that the minimal total turning of a curve in $\hat{\mathcal{L}}^{\kappa_2}_{\kappa_1}(Q)$ increases to infinity as $\kappa_1 > 0$ increases to infinity (for fixed $Q = (q,z)$ with $q \neq 0$).

References

[1] Dan Burghelea, Nicolau C. Saldanha, and Carlos Tomei, *Results on infinite-dimensional topology and applications to the structure of the critical sets of nonlinear Sturm-Liouville operators*, J. of Differential Equations **188** (2003), 569–590.

[2] Lester E. Dubins, *On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents*, Amer. J. Math. **79** (1957), no. 3, 497–516.

[3] ———, *On plane curves with curvature*, Pacific J. Math. **11** (1961), no. 2, 471–481.

[4] David Henderson, *Infinite-dimensional manifolds are open subsets of Hilbert space*, Bull. Amer. Math. Soc. **75** (1969), no. 4, 759–762.

[5] Heinz Hopf, *Zum Clifford-Kleinschen Raumproblem*, Math. Ann. **95** (1926), 313–339.

[6] John A. Little, *Nondegenerate homotopies of curves on the unit 2-sphere*, J. of Differential Geom. **4** (1970), 339–348.

[7] Richard Palais, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.

[8] Nicolau C. Saldanha, *The homotopy type of spaces of locally convex curves in the sphere*, arXiv:1207.4221, 2012.

[9] Nicolau C. Saldanha and Pedro Zühlke, *Spaces of curves with constrained curvature on flat surfaces, II*, in preparation.

[10] ———, *On the components of spaces of curves on the 2-sphere with geodesic curvature in a prescribed interval*, Internat. J. Math. (to appear), arxiv:1304.2629.

[11] Stephen Smale, *Regular curves on Riemannian manifolds*, Trans. Amer. Math. Soc. **87** (1956), no. 2, 492–512.

[12] Hassler Whitney, *On regular closed curves in the plane*, Compos. Math. **4** (1937), no. 1, 276–284.

[13] Laurent Younes, *Shapes and diffeomorphisms*, Springer-Verlag, 2010.

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