The homotopy type of spaces of locally convex curves in the sphere

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Abstract

A smooth curve $\gamma : [0,1] \to \mathbb{S}^2$ is locally convex if its geodesic curvature is positive at every point. J. A. Little showed that the space of all locally convex curves γ with $\gamma(0) = \gamma(1) = e_1$ and $\gamma'(0) = \gamma'(1) = e_2$ has three connected components $\mathcal{L}_{-\mathbf{1},c}$, $\mathcal{L}_{+\mathbf{1}}$, $\mathcal{L}_{-\mathbf{1},n}$. The space $\mathcal{L}_{-\mathbf{1},c}$ is known to be contractible. We prove that $\mathcal{L}_{+\mathbf{1}}$ and $\mathcal{L}_{-\mathbf{1},n}$ are homotopy equivalent to $(\Omega\mathbb{S}^3) \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \cdots$ and $(\Omega\mathbb{S}^3) \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \mathbb{S}^{12} \vee \cdots$, respectively. As a corollary, we deduce the homotopy type of the components of the space $\mathrm{Free}(\mathbb{S}^1, \mathbb{S}^2)$ of free curves $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$ (i.e., curves with nonzero geodesic curvature). We also determine the homotopy type of the spaces $\mathrm{Free}([0,1], \mathbb{S}^2)$ with fixed initial and final frames.

1 Introduction

A curve $\gamma : [0,1] \to \mathbb{S}^2$ is called *locally convex* if its geodesic curvature is always positive, or, equivalently, if $\det(\gamma(t), \gamma'(t), \gamma''(t)) > 0$ for all t. Let $\mathcal{L}_I$ be the space of all locally convex curves γ with $\gamma(0) = \gamma(1) = e_1$ and $\gamma'(0) = \gamma'(1) = e_2$; the precise topology for this space of curves will be discussed in the paper. J. A. Little [15] showed that $\mathcal{L}_I$ has three connected components $\mathcal{L}_{-\mathbf{1},c}$, $\mathcal{L}_{+\mathbf{1}}$, $\mathcal{L}_{-\mathbf{1},n}$; examples of curves in each connected component are shown in Figure 1.

The connected component $\mathcal{L}_{-\mathbf{1},c}$ can be defined to be the set of simple curves in $\mathcal{L}_I$: the space $\mathcal{L}_{-\mathbf{1},c}$ is known to be contractible ([1] and [25], Lemma 5). The aim of this paper is to determine the homotopy type of the two remaining spaces $\mathcal{L}_{+\mathbf{1}}$ and $\mathcal{L}_{-\mathbf{1},n}$. Our main result is the following.

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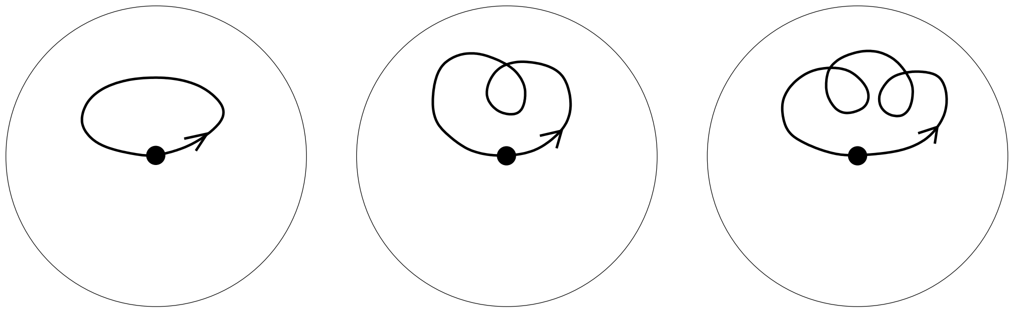

Figure 1: Curves in $\mathcal{L}_{\mathbf{-1},c}$, $\mathcal{L}_{\mathbf{+1}}$ and $\mathcal{L}_{\mathbf{-1},n}$.

Theorem 1 *The components $\mathcal{L}_{\mathbf{+1}}$ and $\mathcal{L}_{\mathbf{-1},n}$ are homotopically equivalent to $(\Omega\mathbb{S}^3) \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \cdots$ and $(\Omega\mathbb{S}^3) \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \mathbb{S}^{12} \vee \cdots$, respectively.*

Here $\Omega\mathbb{S}^3$ is the space of loops in $\mathbb{S}^3$, i.e., the set of continuous maps $\alpha : [0,1] \to \mathbb{S}^3$ with $\alpha(0) = \alpha(1) = \mathbf{1}$, where $\mathbf{1} \in \mathbb{S}^3$ is a base point, with the C^0 topology. A more careful description of the connected components $\mathcal{L}_{\mathbf{+1}}$ and $\mathcal{L}_{\mathbf{-1},n}$ is given below.

A motivation for considering these spaces comes from differential equations. Consider the linear ODE of order 3:

$$u'''(t) + h_1(t)u'(t) + h_0(t)u(t) = 0, \quad t \in [0,1];$$

the set of pairs of potentials (h_0, h_1) for which the equation admits 3 linearly independent periodic solutions is homotopically equivalent to $\mathcal{L}_I$. The corresponding problem in order 2 is much simpler ([5], [6], [21]).

Alternatively, in Gromov's language ([11], [7]), given two smooth Riemannian manifolds V^n and W^q, a map $f : V^n \to W^q$ is *free* (or *second order free*) if the second order osculating space is non-degenerate (we use covariant derivatives); let $\mathrm{Free}(V,W)$ be the space of such free maps. Perhaps the simplest non-trivial example here is $\mathrm{Free}(\mathbb{S}^1, \mathbb{S}^2)$, the space of curves $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$ with $\det(\gamma(t), \gamma'(t), \gamma''(t)) \neq 0$ for all t. Thus $\mathrm{Free}(\mathbb{S}^1, \mathbb{S}^2)$ differs from our space $\mathcal{L}_I$ only by the fact that in $\mathrm{Free}(\mathbb{S}^1, \mathbb{S}^2)$ negative determinants are allowed, as are arbitrary initial frames $Q \in SO_3$. The following result is a direct consequence of Theorem 1.

Corollary 1.1 *The space $\mathrm{Free}(\mathbb{S}^1, \mathbb{S}^2)$ has six connected components, with two homotopically equivalent to each of the following spaces:*

$$SO_3 \times \mathcal{L}_{\mathbf{-1},c} \approx SO_3,$$
$$SO_3 \times \mathcal{L}_{\mathbf{+1}} \approx SO_3 \times \left((\Omega\mathbb{S}^3) \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \cdots\right),$$
$$SO_3 \times \mathcal{L}_{\mathbf{-1},n} \approx SO_3 \times \left((\Omega\mathbb{S}^3) \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \mathbb{S}^{12} \vee \cdots\right).$$

Recall that if $q > \frac{n(n+3)}{2}$ then free maps satisfy the parametrical h-principle; we are here in the critical case $n = 1$ and $q = \frac{n(n+3)}{2} = 2$ and the principle (incorrectly applied) would predict a (wrong) simpler answer. This paper does not require any familiarity with these ideas but the reader may notice that ideas similar to the h-principle will play an important part.

These spaces and variants have been discussed, among others, by B. Shapiro, M. Shapiro and B. Khesin ([24], [23]). These spaces are also the orbits of the second Gel'fand-Dikki brackets and therefore have a natural symplectic structure ([9], [10]). Furthermore, these spaces are related to the orbit classification of the Zamolodchikov Algebra ([13], [17]); these interpretations shall not be used or discussed in this paper.

Although the above authors ponder about the interest of understanding the topology of such spaces, their results deal mostly with π_0, i.e., with counting and identifying connected components. The present author has also previously proved some weaker results about the topology of these spaces, regarding the fundamental group and the first few (co)homology groups ([18], [19]); these results are now of course easy consequences of Theorem 1.

Corollary 1.2 *The spaces $\mathcal{L}_{+\mathbf{1}}$ and $\mathcal{L}_{-\mathbf{1},n}$ are connected and simply connected and, for $k > 0$, their cohomology is given by*

$$H^k(\mathcal{L}_{+\mathbf{1}};\mathbb{Z}) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}^2, & 4 \mid (k+2), \\ \mathbb{Z}, & 4 \mid k; \end{cases} \qquad H^k(\mathcal{L}_{-\mathbf{1},n};\mathbb{Z}) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & 4 \mid (k+2), \\ \mathbb{Z}^2, & 4 \mid k. \end{cases}$$

Let $\mathcal{I}$ be the space of immersions $\gamma : [0,1] \to \mathbb{S}^2$ (of class C^k for some $k \geq 2$) with $\gamma'(t) \neq 0$, $\gamma(0) = e_1$, $\gamma'(0) = e_2$. Let $\mathcal{L} \subset \mathcal{I}$ be the subspace of locally convex curves; thus, for $\gamma \in \mathcal{I}$ we have $\gamma \in \mathcal{L}$ if and only if $\det(\gamma(t), \gamma'(t), \gamma''(t)) > 0$ for all t. For each $\gamma \in \mathcal{I}$, consider its *Frenet frame* $\mathfrak{F}_\gamma : [0,1] \to SO_3$ defined by

$$\begin{pmatrix} \gamma(t) & \gamma'(t) & \gamma''(t) \end{pmatrix} = \mathfrak{F}_\gamma(t) R(t),$$

$R(t)$ being an upper triangular matrix with $(R(t))_{11} > 0$ and $(R(t))_{22} > 0$ (the left hand side is the 3×3 matrix with columns $\gamma(t)$, $\gamma'(t)$ and $\gamma''(t)$). In other words, the first column of $\mathfrak{F}_\gamma(t)$ is $\gamma(t)$, the second is the unit tangent vector $\mathbf{t}_\gamma(t) = \gamma'(t)/|\gamma'(t)|$ and the third column (which is now uniquely determined) is the unit normal vector $\mathbf{n}_\gamma(t) = \gamma(t) \times \mathbf{t}_\gamma(t)$. For $Q \in SO_3$, let $\mathcal{I}_Q \subset \mathcal{I}$ be the set of curves $\gamma \in \mathcal{I}$ for which $\mathfrak{F}_\gamma(1) = Q$; similarly, let $\mathcal{L}_Q = \mathcal{L} \cap \mathcal{I}_Q$.

The universal (double) cover of SO_3 is $\mathbb{S}^3 \subset \mathbb{H}$, the group of quaternions of absolute value 1; let $\Pi : \mathbb{S}^3 \to SO_3$ be the canonical projection. For $\gamma \in \mathcal{I}$, the curve $\mathfrak{F}_\gamma$ can be lifted to define $\tilde{\mathfrak{F}}_\gamma : [0,1] \to \mathbb{S}^3$ with $\tilde{\mathfrak{F}}_\gamma(0) = \mathbf{1}$, $\Pi \circ \tilde{\mathfrak{F}}_\gamma = \mathfrak{F}_\gamma$. The value of $\tilde{\mathfrak{F}}_\gamma(1)$ partitions $\mathcal{I}_Q$ as a disjoint union $\mathcal{I}_z \sqcup \mathcal{I}_{-z}$: here $\Pi(\pm z) = Q$ and $\gamma \in \mathcal{I}_z$ if and only if $\tilde{\mathfrak{F}}_\gamma(1) = z$; similarly, let $\mathcal{L}_z = \mathcal{L}_{\Pi(z)} \cap \mathcal{I}_z$. Notice that if γ is a simple curve in $\mathcal{I}_I$ then $\tilde{\mathfrak{F}}_\gamma(1) = -\mathbf{1}$ and therefore $\gamma \in \mathcal{I}_{-\mathbf{1}}$. We can now more precisely describe the three connected components of $\mathcal{L}_I$: the component $\mathcal{L}_{+\mathbf{1}}$ is a special case of this definition and $\mathcal{L}_{-\mathbf{1}}$ has two connected components $\mathcal{L}_{-\mathbf{1},c}$ (convex or simple curves) and $\mathcal{L}_{-\mathbf{1},n}$ (non-simple).

A locally convex curve $\gamma : [t_0, t_1] \to \mathbb{S}^2$ is *convex* if γ intersects any geodesic (great circle) at most twice. This definition requires a couple of clarifications. First, endpoints do not count as intersections so, for instance, a simple closed locally convex curve is convex. Second, intersections are counted with multiplicity so that a tangency counts as two intersections. It follows from the definition that convex curves are simple. In this paper we will see other equivalent definitions.

A matrix $Q \in SO_3$ is *convex* if there exists a convex arc $\gamma \in \mathcal{L}$ with $\mathfrak{F}_\gamma(1) = Q$. We shall see that the subset of SO_3 of convex matrices is the disjoint union of one of the top dimensional Bruhat cells and a few lower dimensional cells contained in its closure. Similarly, a quaternion $z \in \mathbb{S}^3$ is *convex* if there exists a convex arc $\gamma \in \mathcal{L}$ with $\tilde{\mathfrak{F}}_\gamma(1) = z$. It is not hard to see that if $-z$ is convex then z is not.

We can now state a more general version of our main theorem. Here $\mathbf{i} \in \mathbb{S}^3$ is the usual quaternion. Notice that $-\mathbf{1}$ is convex but $\mathbf{1}$, $\mathbf{i}$ and $-\mathbf{i}$ are not.

Theorem 2 *Let $z \in \mathbb{S}^3$. Then the space $\mathcal{L}_z$ is homotopically equivalent to $\mathcal{L}_{-\mathbf{1}}$ if z is convex, $\mathcal{L}_{\mathbf{1}}$ if $-z$ is convex and $\mathcal{L}_{\mathbf{i}}$ otherwise. Moreover, the following homotopy equivalences hold:*

$$\mathcal{L}_{-\mathbf{1}} \approx (\Omega\mathbb{S}^3) \vee \mathbb{S}^0 \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \cdots; \quad \mathcal{L}_{+\mathbf{1}} \approx (\Omega\mathbb{S}^3) \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \cdots; \quad \mathcal{L}_{\mathbf{i}} \approx \Omega\mathbb{S}^3.$$

We prove Theorem 2 not just for the sake of proving a stronger statement but mainly because it is not clear how to produce a complete proof of Theorem 1 (whose statement is simpler and more natural) without strong use of Bruhat cells and other algebraic notions. By the time these ideas have been mastered, both theorems are proved simultaneously.

Let $\Omega_z\mathbb{S}^3$ be the space of continuous curves $\alpha : [0,1] \to \mathbb{S}^3$, $\alpha(0) = \mathbf{1}$, $\alpha(1) = z$: this is easily seen to be homeomorphic to $\Omega\mathbb{S}^3$ and the two spaces shall from now on be identified. We just constructed the map $\tilde{\mathfrak{F}} : \mathcal{I}_z \to \Omega_z\mathbb{S}^3$, $\gamma \mapsto \tilde{\mathfrak{F}}_\gamma$. It is a well-known fact (which follows from the Hirsch-Smale Theorem) that this map is a homotopy equivalence ([16], [12], [26]). Theorem 2 implies that the inclusions $i : \mathcal{L}_z \to \mathcal{I}_z$ are homotopy equivalences only for certain quaternions z.

We now proceed to give an overview of the paper and of the proof of the main theorems. Section 2 addresses the rather technical issue of what, precisely, is the best topology for the space $\mathcal{L}_z$. As we shall see, we may allow for the juxtaposition of curves (provided their Frenet frames agree); on the other hand, when desirable, we may assume curves to be smooth. In Section 3 we apply the construction of Bruhat cells to our scenario. We also study projective transformations. The short Section 4 collects a few useful facts about the total (euclidean) curvature of spherical curves.

It follows from Little's results that a circle drawn twice and a circle drawn four times are in the same connected component of $\mathcal{L}_I$. In Section 5 we give a careful description of a path joining these two curves. We also prove a few facts about this path which will be needed later.

As we have just mentioned, the inclusion $i : \mathcal{L}_z \to \mathcal{I}_z$ need not be a homotopy equivalence. In Section 6 we see that half of the story, so to speak, still holds.

Proposition 1.3 *Let $z \in \mathbb{S}^3$. For any compact space K and any function $f : K \to \mathcal{I}_z$ there exists $g : K \to \mathcal{L}_z$ and a homotopy $H : [0,1] \times K \to \mathcal{I}_z$ with $H(0,p) = f(p)$ and $H(1,p) = g(p)$ for all $p \in K$.*

The maps $i : \mathcal{L}_z \to \mathcal{I}_z$ therefore induce surjective maps $\pi_k(\mathcal{L}_z) \to \pi_k(\mathcal{I}_z)$.

In a nutshell, if a curve has very large (positive) geodesic curvature it looks like a phone wire and we call it *sufficiently loopy*. Any compact family $\alpha_0 : K \to \mathcal{I}_z$ can be approximated in the C^0 topology by a family $\alpha_1 : K \to \mathcal{I}_z$ of sufficiently loopy (and therefore locally convex) curves (in Figure 2, $\gamma_0 = \alpha_0(p)$ and $\gamma_1 = \alpha_1(p)$ for some $p \in K$). Also, families of sufficiently loopy curves can be deformed without losing the property of being locally convex.

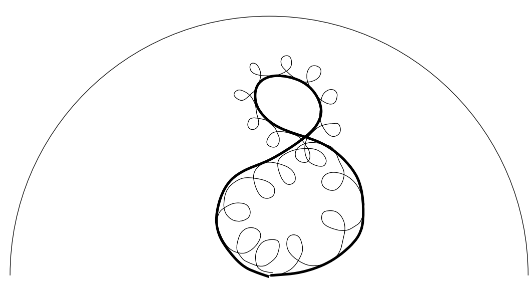

Figure 2: Curves $\gamma_0 \in \mathcal{I}_{\pm\mathbf{1}}$ (thick) and $\gamma_1 \in \mathcal{L}_{\pm\mathbf{1}}$ (thin).

The difficulty in proving (the false fact) that the inclusion $\mathcal{L}_z \subset \mathcal{I}_z$ is a homotopy equivalence is that there is no *uniform* procedure to add loops to locally convex curves within the set of locally convex curves.

In Section 7 we introduce a crucial construction in our discussion: a curve $\gamma \in \mathcal{L}_Q$ is *multiconvex* of multiplicity k if it is the juxtaposition of $k-1$ simple closed convex curves with a final k-th convex curve in $\mathcal{L}_Q$ (see Figure 3).

We prove in Lemma 7.1 that, given Q, the closed subset $\mathcal{M}_k \subset \mathcal{L}_Q$ of multiconvex curves of multiplicity k is either empty or a contractible submanifold

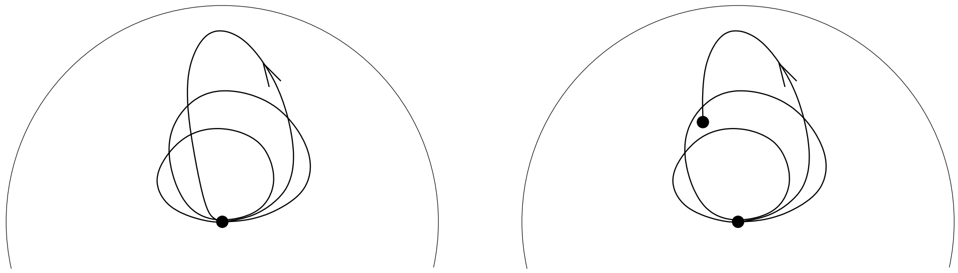

Figure 3: Two multiconvex curves of multiplicity 3

of codimension $2k-2$ with trivial normal bundle. Assuming $-z$ convex, we next construct in Lemma 7.3 maps $h_{2k-2} : \mathbb{S}^{2k-2} \to \mathcal{L}_{(-\mathbf{1})^k z}$ which intersect $\mathcal{M}_k$ transversally and exactly once and are homotopic to a constant as maps $\mathbb{S}^{2k-2} \to \mathcal{I}_{(-\mathbf{1})^k z}$ (details of the construction of the path in Section 5 are used here to verify that h_{2k-2} has the desired properties). Intersection with $\mathcal{M}_k$ shows that h_{2k-2} is not homotopic to a constant as a map $\mathbb{S}^{2k-2} \to \mathcal{L}_I$ and therefore defines a non-trivial element of the homotopy group $\pi_{2k-2}(\mathcal{L}_{(-\mathbf{1})^k z})$ which is taken to zero by the inclusion in $\mathcal{I}_{(-\mathbf{1})^k z}$. Furthermore, intersection with $\mathcal{M}_k$ defines an element of the cohomology group $H^{2k-2}(\mathcal{L}_{(-\mathbf{1})^k z})$ not in the image of $i^* : H^*(\mathcal{I}_{(-\mathbf{1})^k z}) \to H^*(\mathcal{L}_{(-\mathbf{1})^k z})$ (compare with [18] and [19]). This does not prove our main theorem yet but already shows that if either z or $-z$ is convex then $\mathcal{L}_z$ is not homotopically equivalent to $\mathcal{I}_z$.

In Section 8 we introduce *grafting*, a process under which loops can sometimes be added to curves. In Section 9 we define the *next step* function and learn to tell apart *good* and *bad steps*; Bruhat cells are essential here. Let $\mathcal{Y}_z = \mathcal{L}_z \smallsetminus \bigcup_k \mathcal{M}_k$ be the set of *complicated* (i.e., not multiconvex) curves. The new tools introduced in Sections 8 and 9 are then used in Section 10 to understand the spaces $\mathcal{Y}_z$.

Proposition 1.4 *The inclusion $\mathcal{Y}_z \subset \mathcal{I}_z$ is a weak homotopy equivalence.*

In order to determine the homotopy type of $\mathcal{L}_z$, start with $\mathcal{Y}_z$ and, for each k, add the set $\mathcal{M}_k$. It follows from what we have proved that adding $\mathcal{M}_k$ (if nonempty) is equivalent to attaching a sphere $\mathbb{S}^{2k-2}$. These properties are then sufficient to complete, in Section 11, the proof of Theorems 1 and 2.

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2 Topology of $\mathcal{L}_Q$

We now attack a rather technical problem: defining the best topological structure for the spaces $\mathcal{L}_Q$ and $\mathcal{I}_Q$.

It turns out that different topological structures obtain different spaces which are however homotopically equivalent. The C^k metric for some $k \geq 2$ is a rather natural choice but it has the inconvenience that when constructing a homotopy we prefer not to be distracted by the necessity to smoothen out certain points of our curves (we want to be allowed, for instance, to consider a curve γ which is a juxtaposition of arcs of circle).

Given a smooth immersion $\gamma : [0,1] \to \mathbb{S}^2$, let $\gamma'(t) = v_\gamma(t)\mathbf{t}_\gamma(t)$ where $v_\gamma(t) = |\gamma'(t)| > 0$ and $\mathbf{t}_\gamma(t)$ is the unit tangent vector to γ. Let $\mathbf{n}_\gamma(t) = \gamma(t) \times \mathbf{t}_\gamma(t)$ be the unit normal vector to γ, so that

$$\mathbf{t}'_\gamma(t) = -v_\gamma(t)\gamma(t) + \hat{v}_\gamma(t)\mathbf{n}_\gamma(t), \quad \mathbf{n}'_\gamma(t) = -\hat{v}_\gamma(t)\mathbf{t}_\gamma(t)$$

where $\hat{v}_\gamma(t) = \kappa_\gamma(t)v_\gamma(t)$, $\kappa_\gamma(t)$ being the geodesic curvature of γ. The curve γ is locally convex if and only if $\hat{v}_\gamma(t) > 0$ for all t. In matrix notation, $\gamma(t)$, $\mathbf{t}_\gamma(t)$ and $\mathbf{n}_\gamma(t)$ are the columns of the orthogonal matrix $\mathfrak{F}_\gamma(t)$ which satisfies

$$\mathfrak{F}'_\gamma(t) = \mathfrak{F}_\gamma(t)\Lambda_\gamma(t); \quad \Lambda_\gamma(t) = \begin{pmatrix} 0 & -v_\gamma(t) & 0 \\ v_\gamma(t) & 0 & -\hat{v}_\gamma(t) \\ 0 & \hat{v}_\gamma(t) & 0 \end{pmatrix}.$$

Let $V \subset so_3$ be the plane (i.e., 2-dimensional real vector space) of matrices M with $(M)_{31} = 0$; Λ_γ can be considered a function from $[0,1]$ to V. Let $V_\mathcal{I} \subset V$ be the half-plane $(M)_{21} > 0$; Λ_γ can also be considered a function $\Lambda_\gamma : [0,1] \to V_\mathcal{I}$. Conversely, given a smooth function $\Lambda_\gamma : [0,1] \to V_\mathcal{I}$ or, equivalently, v_γ and $\hat{v}_\gamma$, the above equations together with $\mathfrak{F}_\gamma(0) = I$ may be interpreted as an initial value problem defining $\mathfrak{F}_\gamma(t)$ and therefore $\gamma(t) = \mathfrak{F}_\gamma(t)e_1$.

Our aim is to consider a reasonably large space of functions which still allows the initial value problem above to be solved. The Hilbert space $L^2([0,1])$ is now a natural choice: for $v, \hat{v} \in L^2([0,1])$ the initial value problem can be solved. We therefore interpret $\mathcal{I}_Q$ to be the closed subset of $(L^2([0,1]))^2$ of pairs of functions $(w, \hat{v})$ such that the solution $\Gamma : [0,1] \to SO_3$ of the initial value problem

$$\Gamma'(t) = \Gamma(t)\Lambda(t), \quad \Gamma(0) = I, \tag{1}$$

satisfies $\Gamma(1) = Q$, where

$$\Lambda(t) = \begin{pmatrix} 0 & -v(t) & 0 \\ v(t) & 0 & -\hat{v}(t) \\ 0 & \hat{v}(t) & 0 \end{pmatrix}, \quad v(t) = \frac{w(t) + \sqrt{(w(t))^2 + 4}}{2}$$

(we have $w(t) = v(t) - 1/v(t)$, $v(t) > 0$). The fact that $\Lambda(t)$ belongs to the Lie algebra so_3 guarantees that $\Gamma(t)$ assumes values in the corresponding Lie group SO_3. Also, the function Γ defined above is continuous, absolutely continuous and differentiable almost everywhere. In this way $\mathcal{I}_Q \subset (L^2([0,1]))^2$ is a smooth Hilbert manifold of codimension 3. Indeed, consider the map $\omega_{\mathcal{I}} : (L^2([0,1]))^2 \to SO_3$ taking $(w, \hat{v})$ to $\Gamma(1)$, where $\Gamma : [0,1] \to SO_3$ is defined by the initial value problem (1) above. Smooth dependence on parameters tells us that this map is smooth; let us compute its derivative.

In general, for a curve $\Gamma : [0,1] \to SO_3$, write $\Gamma(t_0; t_1) = (\Gamma(t_0))^{-1}\Gamma(t_1)$. Let L be a one-parameter family of functions $L(s) : [0,1] \to V_{\mathcal{I}}$, $s \in (-\epsilon, \epsilon)$ with $L(0) = \Lambda$. Let $G(s) : [0,1] \to SO_3$ be the solution of

$$(G(s))'(t) = (G(s))(t)(L(s))(t), \quad (G(s))(0) = I$$

so that $G(0) = \Gamma$; notice that the derivative in this initial value problem is with respect to t. An easy computation gives that the derivative of G (with respect to s) satisfies

$$(G'(s))(t)(G(s)(t))^{-1} = \int_0^t (G(s))(\tau)(L'(s))(\tau)((G(s))(\tau))^{-1} d\tau.$$

Assume, for instance, that $(L'(0))(t)$ consists of three smooth narrow bumps around times t_i so that

$$(\Gamma(1))^{-1}(G'(0))(t) \approx \sum_i (\Gamma(t_i; 1))^{-1}(L'(0))(t_i)\Gamma(t_i; 1).$$

An easy computation shows that the spaces $(\Gamma(t_i; 1))^{-1} V \Gamma(t_i; 1) \subset so_3$ are not constant and therefore $(\Gamma(1))^{-1}(G'(0))(t)$ may assume any value in so_3. The derivative of $\omega_{\mathcal{I}}$ is therefore surjective. The map $\omega_{\mathcal{I}}$ is thus a submersion and $\mathcal{I}_Q$ a regular level set. The geometric description of $\mathcal{I}_Q$ comes from the identification $(w, \hat{v}) \leftrightarrow \gamma$, where $\gamma(t) = \Gamma(t)e_1$.

Similarly, let $V_{\mathcal{L}} \subset V \subset so_3$ be the quarter-plane $(M)_{31} = 0$, $(M)_{32} > 0$, $(M)_{21} > 0$. If γ is locally convex then the image of Λ_γ is contained in $V_{\mathcal{L}}$ and, conversely, given a smooth function $\Lambda : [0,1] \to V_{\mathcal{L}}$, equation (1) obtains a locally convex curve. Define $\mathcal{L}_Q \subset (L^2([0,1]))^2$ to be the set of pairs $(w, \hat{w})$ such that $(w, \hat{v}) \in \mathcal{I}_Q$ where

$$\hat{v}(t) = \frac{\hat{w}(t) + \sqrt{(\hat{w}(t))^2 + 4}}{2}.$$

As above, define $\omega : (L^2([0,1]))^2 \to SO_3$ taking $(w, \hat{w})$ to $\Gamma(1)$, where Γ is again defined by equation (1). The computation above also shows that the smooth map ω is a submersion and $\mathcal{L}_Q$ is a smooth Hilbert submanifold of codimension 3 in $(L^2([0,1]))^2$. We have a natural injective map $\mathcal{L}_Q \to \mathcal{I}_Q$ taking $(w, \hat{w})$ to $(w, \hat{v})$.

In the spirit of considering $\mathcal{L}_Q$ and $\mathcal{I}_Q$ to be sets of curves we call this map an inclusion (even though it is not an isometry with the above metric).

The space $\mathcal{L}_Q$ we just defined is rather large. The space $\mathcal{L}_Q^{[C^k]}$ of locally convex curves of class C^k ($k \geq 2$) is now naturally identified to a dense subset $\mathcal{L}_Q^{[C^k]} \subset \mathcal{L}_Q$ with a different topology. But are these two spaces similar? Or, more precisely, is the inclusion a weak homotopy equivalence? The answer here is yes.

In order to see this, first consider $\mathcal{L}_Q^{[[C^k]]}$, $k \geq 0$, the subset of $(C^k([0,1]))^2$ of pairs $(w, \hat{w})$ such that $\Gamma(1) = Q$ (where Γ is defined by the initial value problem (1)). The inclusion $\mathcal{L}_Q^{[[C^k]]} \subset \mathcal{L}_Q$ is a homotopy equivalence: this follows directly from Theorem 2 from [4]. For the convenience of the reader, we quote here a simplified version of that result.

Proposition 2.1 *Let X and Y be separable Banach spaces. Suppose $i : Y \to X$ is a bounded, injective linear map with dense image and $M \subset X$ a smooth, closed submanifold of finite codimension. Then $N = i^{-1}(M)$ is a smooth closed submanifold of Y and the restriction $i : N \to M$ is a homotopy equivalence.*

For $k \geq 1$ the curves γ are now of class C^2 and we may assume them to be parametrized by a constant multiple of arc length (this does not change the homotopy type of the space since the group of orientation-preserving diffeomorphisms of $[0,1]$ is contractible). In terms of the pairs $(w, \hat{w})$, this says that we may assume w to be constant. But for w constant, $\hat{w}$ is of class C^k if and only if γ is of class C^{k+2}, completing the argument.

Summing up, and simplifying this discussion a little, we may assume our curves γ to be as smooth as our constructions require but when constructing a homotopy we may use curves for which the curvature is only piecewise continuous. We will however try to be as consistent as reasonably possible in the use of the large space $\mathcal{L}$ defined above.

3 Projective transformations and Bruhat cells

In this section we present some more algebraic notation, especially the decomposition of SO_3 and $\mathbb{S}^3$ in (signed) *Bruhat cells*. Some of this material is presented in [20] in a more algebraic fashion and for arbitrary dimension.

The *projective transformation* $\pi(A) : \mathbb{S}^2 \to \mathbb{S}^2$ associated to $A \in SL_3$ is defined by $\pi(A)(v) = \widehat{Av} = (1/|Av|)Av$. Similarly, define $\pi(A) : SO_3 \to SO_3$ by $\pi(A)(Q_0) = Q_1$ if there exists $U_1 \in Up_3^+$ with $AQ_0 = Q_1 U_1$; here Up_3^+ is the contractible group of upper triangular matrices with positive diagonal and determinant $+1$. In other words, $\pi(A)(Q_0)$ is obtained from AQ_0 by performing Gram-Schmidt on columns. In particular, if $A \in SO_3$ then $\pi(A)(Q_0) = AQ_0$.

This action of SL_3 on SO_3 is transitive but not doubly transitive; we shall soon discuss the extent to which it fails to be doubly transitive.

Notice that $\pi(A) : \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation preserving diffeomorphism. For an immersion $\gamma : [0,1] \to \mathbb{S}^2$ define $\pi(A)\gamma$ by $(\pi(A)\gamma)(t) = \pi(A)(\gamma(t))$; the curve $\pi(A)\gamma$ is again an immersion. Since $\pi(A)$ takes geodesics to geodesics, if $\gamma : [0,1] \to \mathbb{S}^2$ is locally convex then so is $\pi(A)\gamma$ (this can also be checked directly from the definition by a straightforward computation). The map $\pi(A)$ is defined so that $\mathfrak{F}_{\pi(A)\gamma} = \pi(A)(\mathfrak{F}_\gamma)$ for any immersion $\gamma : [0,1] \to \mathbb{S}^2$. Notice that $\pi(A)I = I$ if and only if $A \in \mathit{Up}_3^+$. Thus, for $U \in \mathit{Up}_3^+$, $\pi(U)$ defines a smooth map from $\mathcal{L}_Q$ to $\mathcal{L}_{\pi(U)Q}$.

Let $B_3 \subset O_3$ be the Coxeter-Weyl group of signed permutation matrices: the group B_3 has 48 elements and corresponds to the isometries of the octahedron of vertices $\pm e_1, \pm e_2, \pm e_3$. Let $B_3^+ = B_3 \cap SO_3$ be the subgroup of orientation preserving isometries. Dropping signs defines a homomorphism from B_3^+ to the symmetric group S_3: the *number of inversions* of $Q \in B_3^+$ is the number of inversions of the corresponding permutation in S_3 (recall that the number of inversions of $\pi \in S_n$ is the number of pairs (i,j) with $i < j$ and $\pi(i) > \pi(j)$). We denote an element of B_3^+ by the letter P (for permutation), indicating in the subscript the corresponding permutation (as a cycle) and the signs, read by column, in binary:

$$0 = +++,\ 1 = ++-,\ 2 = +-+,\ 3 = +--,\ \ldots,\ 7 = ---;$$

thus, for instance

$$P_{(13);1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & +1 & 0 \\ +1 & 0 & 0 \end{pmatrix}, \quad P_{(13);2} = \begin{pmatrix} 0 & 0 & +1 \\ 0 & -1 & 0 \\ +1 & 0 & 0 \end{pmatrix}.$$

The lift $\tilde{B}_3^+ \subset \mathbb{S}^3$ of B_3^+ is the group of 48 quaternions with either one coordinate of absolute value 1 and three equal to 0 or two coordinates of absolute value $1/\sqrt{2}$ and two equal to 0 or four coordinates of absolute value $1/2$. As a subset of $\mathbb{R}^4$, $\tilde{B}_3^+$ is also the root system F_4 (with all vectors of size 1, unlike what is usual for Lie algebras). For instance,

$$\Pi\left(\frac{\mathbf{1}-\mathbf{j}}{\sqrt{2}}\right) = P_{(13);1}, \quad \Pi\left(\frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}}\right) = P_{(13);2}.$$

The Bruhat cell $\mathrm{Bru}(Q) \subset SO_3$ of $Q \in SO_3$ is the set of all orthogonal matrices of the form $U_0 Q U_1^{-1}$, with $U_0, U_1 \in \mathit{Up}_3^+$. Each Bruhat cell contains a unique element of B_3^+. We also denote the Bruhat cells by Bru_*, with subscripts defined as for $P_* \in B_3^+$; thus, for instance, $\mathrm{Bru}(P_{(13);1}) = \mathrm{Bru}_{(13);1}$. The cell $\mathrm{Bru}(P)$ ($P \in B_3^+$) is diffeomorphic to $\mathbb{R}^d$, where d is the number of inversions of

P. There are therefore exactly four open cells, corresponding to the two matrices above plus

$$\Pi\left(\frac{\mathbf{1}+\mathbf{j}}{\sqrt{2}}\right) = P_{(13);4}, \quad \Pi\left(\frac{\mathbf{i}-\mathbf{k}}{\sqrt{2}}\right) = P_{(13);7}.$$

If Q_0 and Q_1 belong to the same Bruhat cell then there exists a canonical and explicit diffeomorphism between the Hilbert manifolds $\mathcal{L}_{Q_0}$ and $\mathcal{L}_{Q_1}$. Indeed, for Q_0 and $Q_1 \in \mathrm{Bru}(Q_0)$ let $U_0, U_1 \in Up_3^+$ be such that $U_0 Q_0 = Q_1 U_1$; U_0 is uniquely defined if we require $U_0 \in Up_3^1$, where $Up_3^1 \subset Up_3^+$ is the subgroup of matrices with diagonal entries equal to $+1$. Then the map $\pi(U_0) : \mathcal{L}_{Q_0} \to \mathcal{L}_{Q_1}$ is a diffeomorphism; its inverse is the similarly constructed map $\pi(U_0^{-1})$.

Figure 4 shows example of curves $\gamma \in \mathcal{L}_Q$ for $Q \in \mathrm{Bru}_{(13);\ell}$ for $\ell = 1, 2, 4, 7$ (in this order).

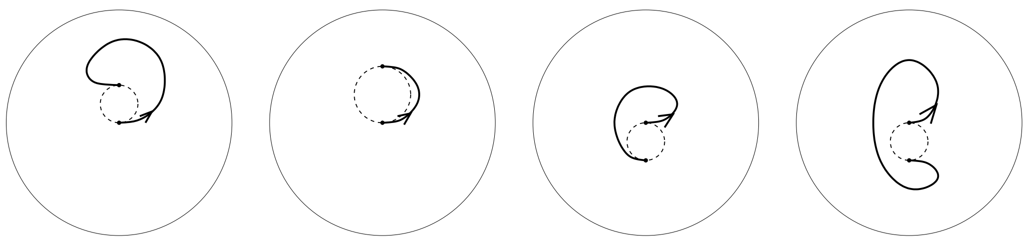

Figure 4: Representatives of the open Bruhat cells

The dashed closed convex curves indicate a convenient geometric way to recognize these Bruhat cells: in all cases there exists a closed convex curve tangent to both endpoints of γ and orientations at the endpoints allow us to distinguish between cells.

For $\gamma \in \mathcal{L}$, let $\Gamma = \mathfrak{F}_\gamma : [0,1] \to SO_3$ and write $\mathfrak{F}_\gamma(t_0; t) = \Gamma(t_0; t) = (\Gamma(t_0))^{-1}\Gamma(t)$. Similarly, $\tilde{\mathfrak{F}}_\gamma(t_0; t) = \tilde{\Gamma}(t_0; t) = (\tilde{\Gamma}(t_0))^{-1}\tilde{\Gamma}(t)$ where $\tilde{\Gamma} : [0,1] \to \mathbb{S}^3$ is a lift of Γ. Clearly, $\Gamma(0; t) = \Gamma(t)$, $\Gamma(t_0; t_0) = I$, $\tilde{\Gamma}(t_0; t_0) = \mathbf{1}$. On the other hand, there exists $\epsilon > 0$ such that for all $t \in (t_0, t_0 + \epsilon)$ we have $\Gamma(t_0; t) \in \mathrm{Bru}_{(13),2}$.

A locally convex arc $\gamma|_{[t_0,t_1]}$ is *convex* if $\mathfrak{F}_\gamma(t_0; t) \in \mathrm{Bru}_{(13);2}$ for all $t \in (t_0, t_1)$. Notice that we do not require that $\mathfrak{F}_\gamma(t_0; t_1) \in \mathrm{Bru}_{(13);2}$; if this also happens then γ is *stably convex*. There are five other Bruhat cells to which $\mathfrak{F}_\gamma(t_0; t_1)$ may belong: $\mathrm{Bru}_{(123);6}$, $\mathrm{Bru}_{(132);0}$, $\mathrm{Bru}_{(23);2}$, $\mathrm{Bru}_{(12);4}$ and $\mathrm{Bru}_{e;0} = I$. Examples of convex arcs corresponding to these five cells are given in Figure 5. We shall come back to these five cells again and again.

Recall that a matrix $Q \in SO_3$ is (stably) *convex* if there exists a (stably) convex arc $\gamma : [t_0, t_1] \to \mathbb{S}^2$ with $\mathfrak{F}_\gamma(t_0; t_1) = Q$. Thus, the set of stably convex matrices is the open cell $\mathrm{Bru}_{(13);2}$ and the set of convex matrices is the union of the six Bruhat cells above (including the open cell). Also, a quaternion $z \in \mathbb{S}^3$

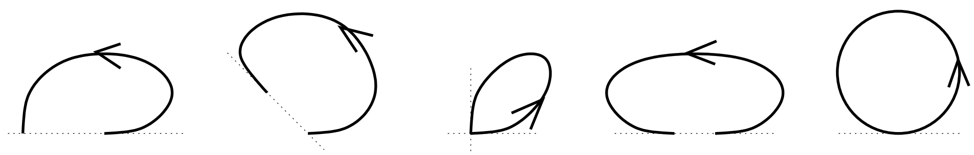

Figure 5: Convex arcs

is (stably) *convex* if there exists a (stably) convex arc $\gamma : [t_0, t_1] \to \mathbb{S}^2$ with $\tilde{\mathfrak{F}}_\gamma(t_0; t_1) = z$. The six convex elements of $\tilde{B}_3^+$ are

$$\frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}}, \quad \frac{-\mathbf{1}+\mathbf{i}-\mathbf{j}+\mathbf{k}}{2}, \quad \frac{-\mathbf{1}+\mathbf{i}+\mathbf{j}+\mathbf{k}}{2}, \quad \frac{-\mathbf{1}+\mathbf{i}}{\sqrt{2}}, \quad \frac{-\mathbf{1}+\mathbf{k}}{\sqrt{2}}, \quad -\mathbf{1};$$

only the first one is stably convex. A quaternion z is (stably) *anticonvex* if $-z$ is (stably) convex. The sets of convex and anticonvex quaternions are unions of distinct Bruhat cells and therefore disjoint; furthermore, the only points of intersection between their respective closures are $\pm\mathbf{1}$.

Let $\mathcal{C}_\nu$ be the circle (contained in $\mathbb{S}^2$) with diameter e_1e_3, parametrized by $\nu_1 \in \mathcal{L}_I$,

$$\nu_1(t) = \frac{1}{2}\left(1+\cos(2\pi t), \sqrt{2}\,\sin(2\pi t), 1-\cos(2\pi t)\right)$$

for which

$$\tilde{\mathfrak{F}}_{\nu_1}(t) = \exp\left(\pi t\,\hat{\mathbf{k}}\right), \quad \tilde{\Lambda}_{\nu_1}(t) = \pi\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} = \frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}},$$

where $\tilde{\Lambda}_\gamma(t) = (\tilde{\mathfrak{F}}_\gamma(t))^{-1}\tilde{\mathfrak{F}}'_\gamma(t)$. For a positive real number s, let $\nu_s(t) = \nu_1(st)$ so that $\nu_s \in \mathcal{L}_{\exp(\pi s\hat{\mathbf{k}})}$. In particular, for integer $k > 0$, $\nu_k \in \mathcal{L}_{(-\mathbf{1})^k}$. We also have $\nu_1 \in \mathcal{L}_{-\mathbf{1},c}$ and $\nu_k \in \mathcal{L}_{-\mathbf{1},n}$ for k odd, $k > 1$.

More generally, a circle of radius $\rho < \pi/2$ is a closed convex curve:

$$\gamma(t) = \cos(2\pi t)\sin(\rho)v_1 + \sin(2\pi t)\sin(\rho)v_2 + \cos(\rho)v_3, \quad t \in [0,1],$$

where v_1, v_2, v_3 is a positively oriented orthonormal basis. Thus, ν_s is a circle of radius $\pi/4$ (measured along the sphere).

The image of a convex circle by a projective transformation is a *spherical ellipse*, or just *ellipse*. Notice that for us an ellipse is an oriented curve. Also, a *projective arc-length parametrization* of an ellipse is a locally convex curve $\gamma : [t_0, t_1] \to \mathbb{S}^2$ of the form $\gamma = \pi(A) \circ \tilde{\gamma}$ where $A \in SL_3$ and $\tilde{\gamma}$ is a parametrization by a multiple of arc length of a circle. We shall sometimes use ellipses when a concrete choice of convex arc is desirable.

Lemma 3.1 *Let $\gamma : [0,1] \to \mathbb{S}^2$ be a stably convex arc and let $t \in (0,1)$. Set*

$$\tilde{\mathfrak{F}}_\gamma(0) = z_0, \quad \gamma(t) = v_t, \quad \tilde{\mathfrak{F}}_\gamma(1) = z_1.$$

Then there exists $\epsilon > 0$ such that if $\hat{z}_0 \in \mathbb{S}^3$, $\hat{v}_t \in \mathbb{S}^2$ and $\hat{z}_1 \in \mathbb{S}^3$ are such that

$$|\hat{z}_0 - z_0| < \epsilon, \quad |\hat{v}_t - v_t| < \epsilon, \quad |\hat{z}_1 - z_1| < \epsilon$$

then there exists a unique ellipse $\hat{\mathcal{E}} \subset \mathbb{S}^2$ and projective arc-length parametrization $\hat{\gamma} : [0,1] \to \hat{\mathcal{E}} \subset \mathbb{S}^2$ with

$$\tilde{\mathfrak{F}}_{\hat{\gamma}}(0) = \hat{z}_0, \quad \hat{\gamma}(t) = \hat{v}_t, \quad \tilde{\mathfrak{F}}_{\hat{\gamma}}(1) = \hat{z}_1.$$

Proof: Let $Q_i = \Pi(z_i)$; draw in $\mathbb{S}^2$ geodesics ℓ_i perpendicular to $Q_i e_3$, oriented by $Q_i e_2$ at $Q_i e_1$ so that ℓ_i is tangent to γ at $t = i$ ($i = 0,1$). Since $z_0^{-1} z_1 \in \mathrm{Bru}_{(13);2}$, the geodesics ℓ_0 and ℓ_1 are transversal and divide the sphere into four open regions, with the image of γ contained in the region characterized by being to the left of both ℓ_0 and ℓ_1. Since all the relevant conditions are open, for sufficiently small $\epsilon > 0$ the corresponding geodesics $\hat{\ell}_i$ are also transversal and $\hat{v}_t$ lies to the left of both $\hat{\ell}_0$ and $\hat{\ell}_1$. There exists $A \in SL_3$ such that the projective transformation $\pi(A)$ satisfies $\pi(A)\hat{Q}_0 = I$ and $\pi(A)\hat{Q}_1 = P_{(13);2}$; for sufficiently small $\epsilon > 0$ we may furthermore assume that $\pi(A)\hat{v}_t$ lies in the first octant. There is $c > 0$ such that, for

$$B_c = \begin{pmatrix} c & 0 & 0 \\ 0 & c^{-2} & 0 \\ 0 & 0 & c \end{pmatrix},$$

$\pi(B_c A)\hat{v}_t$ lies on the arc of circle ν_1 defined above, thus obtaining the desired arc of ellipse. Uniqueness follows from the fact that five points determine a conic and three points determine a projective transformation in the line. ■

Recall that two smooth curves *osculate* each other at a common point if they are tangent and have the same curvature at that point. The next lemma may be considered a limit case of the previous one.

Lemma 3.2 *Let $Q \in \mathrm{Bru}_{(13);2}$; then there exists a unique ellipse $\mathcal{E} \subset \mathbb{S}^2$ and projective arc-length parametrization $\gamma : [0,1] \to \mathcal{E} \subset \mathbb{S}^2$ with γ convex, $\gamma(0) = e_1$, $\gamma'(0) = e_2$, $\mathcal{E}$ osculating the circle $\mathcal{C}_{\hat{\mathbf{k}}}$ at e_1 and $\mathfrak{F}_\gamma(1) = Q$.*

Proof: As in the previous proof, there exists $A \in SL_3$ with $\pi(A)I = I$ and $\pi(A)Q = P_{(13);2}$. With B_c as in the previous proof, the ellipses $\gamma_c = \pi(A^{-1}B_c) \circ \nu_1$ satisfy $\gamma_c(0) = e_1$, $\gamma_c'(0) = e_2$, and $\mathfrak{F}_{\gamma_c}(1) = Q$; there exists a unique $c > 0$ for which γ_c oscullates $\mathcal{C}_{\hat{\mathbf{k}}}$. Uniqueness again follows from five points determining a conic (oscullation counts as three points). ■

4 Total curvature

Given a locally convex curve $\gamma : [a,b] \to \mathbb{S}^2$, let $\kappa_\gamma : [a,b] \to \mathbb{R}$ be the geodesic curvature of γ. This must not be confused with the euclidean curvature κ_γ^E of γ interpreted as a curve in $\mathbb{R}^3$: $\kappa_\gamma^E(t) = \sqrt{1 + \kappa_\gamma^2(t)}$. The total (euclidean) curvature of $\gamma : [a,b] \to \mathbb{S}^2$ is

$$\operatorname{tot}(\gamma) = \int_{[a,b]} \kappa_\gamma^E(t)\, |\gamma'(t)| dt.$$

Notice that $\operatorname{tot}(\nu_s) = 2\pi s$.

Lemma 4.1 *Let $\gamma : [a,b] \to \mathbb{S}^2$ be a locally convex curve.*

(a) The total curvature of γ equals the total variation of $\mathbf{t}_\gamma$ and twice the length of $\tilde{\mathfrak{F}}_\gamma$:

$$\operatorname{tot}(\gamma) = \int_{[a,b]} |\mathbf{t}_\gamma'(t)|\, dt = 2 \int_{[a,b]} |\tilde{\mathfrak{F}}_\gamma'(t)|\, dt.$$

(b) If $\gamma \in \mathcal{L}_I$ is a closed convex curve then $\operatorname{tot}(\gamma) \in [2\pi, 4\pi)$.

(c) If $\gamma \in \mathcal{L}$ is a convex arc then $\operatorname{tot}(\gamma) \in (0, 4\pi]$.

(d) If $\operatorname{tot}(\gamma) < \pi$ then γ is convex.

The inequalities above are not necessarily the best possible.

Proof: Item (a) is a straightforward computation. The first inequality in item (b) is the well known general fact that the total curvature of a closed curve is at least 2π. Alternatively, given (a), $\tilde{\Gamma} = \tilde{\mathfrak{F}}_\gamma : [0,1] \to \mathbb{S}^3$ satisfies $\tilde{\Gamma}(0) = \mathbf{1}$, $\tilde{\Gamma}(1) = -\mathbf{1}$ so of course the length of $\tilde{\Gamma}$ must be at least π. For the second inequality in (b), first notice that $\kappa_\gamma^E(t) \le 1 + \kappa_\gamma(t)$ so we must prove that

$$I_1 + I_2 < 4\pi, \quad I_1 = \int_{[0,1]} |\gamma'(t)| dt, \quad I_2 = \int_{[0,1]} \kappa_\gamma(t)\, |\gamma'(t)| dt.$$

By Gauss-Bonnet, $I_2 = 2\pi - A_\gamma < 2\pi$, where A_γ is the area of the smaller disk in $\mathbb{S}^2$ bounded by γ. Let $\mathbf{n}_\gamma : [0,1] \to \mathbb{S}^2$ be the unit normal vector: $\mathbf{n}_\gamma(t) = \mathfrak{F}_\gamma(t) e_3$. A straightforward computation shows that

$$|\mathbf{n}_\gamma'(t)| = \kappa_\gamma(t) |\gamma'(t)|, \quad \kappa_{\mathbf{n}_\gamma}(t) = 1/\kappa_\gamma(t).$$

Thus, again by Gauss-Bonnet,

$$I_1 = \int_{[0,1]} \kappa_{\mathbf{n}_\gamma}(t)\, |\mathbf{n}_\gamma'(t)| dt = 2\pi - A_{\mathbf{n}_\gamma} < 2\pi$$

where $A_{\mathbf{n}_\gamma}$ is (of course) the area of the smaller disk in $\mathbb{S}^2$ bounded by $\mathbf{n}_\gamma$. This completes the proof of (b).

For item (c), consider a convex arc $\gamma : [0,1] \to \mathbb{S}^2$. For arbitrarily small $\epsilon > 0$, $\gamma|_{[0,1-\epsilon]}$ can be extended to a closed convex curve $\tilde{\gamma}$. Thus, from (b), $\operatorname{tot}(\gamma|_{[0,1-\epsilon]}) < \operatorname{tot}(\tilde{\gamma}) < 4\pi$. Since this estimate holds for all ϵ, $\operatorname{tot}(\gamma) \leq 4\pi$.

For item (d), assume that $\gamma_1 : [a,b] \to \mathbb{S}^2$ is convex but that if the domain of γ_1 is extended to define γ_2 then γ_2 is not convex. In other words, γ_1 is similar to one of the curves in Figure 5. A case by case analysis shows that there exist $t_0 < t_1 \in [a,b]$ with $\mathbf{t}_{\gamma_1}(t_1) = -\mathbf{t}_{\gamma_1}(t_0)$ and therefore

$$\operatorname{tot}(\gamma_1) \geq \int_{[t_0,t_1]} |\mathbf{t}'_{\gamma_1}|(t)\, dt \geq \pi.$$

■

5 The map g_0

The aim of this section is to construct a map $g_0 : \mathbb{S}^2 \to \mathcal{L}_{\mathbf{1}} \subset \mathcal{I}_{\mathbf{1}}$. We shall have $g_0(\mathbf{s}) = \nu_2$ and $g_0(\mathbf{n}) = \nu_4$, where $\mathbf{n} = (0,0,+1)$ and $\mathbf{s} = (0,0,-1)$ are the north and south pole, respectively. The existence of such a map proves that ν_2 and ν_4 are in the same connected component of $\mathcal{L}_{\mathbf{1}}$, consistently with Little's result that $\mathcal{L}_I$ has three connected components $\mathcal{L}_{-\mathbf{1},c}$, $\mathcal{L}_{\mathbf{1}}$ and $\mathcal{L}_{-\mathbf{1},n}$ (see Figure 1). As we shall see in Proposition 5.3, g_0 turns out to be a generator of $\pi_2(\mathcal{I}_{\mathbf{1}}) \approx H_2(\mathcal{I}_{\mathbf{1}};\mathbb{Z}) \approx \mathbb{Z}$. The map g_0 is one the the crucial objects in this paper so its construction shall be discussed in some detail.

The map g_0 is shown in Figure 6: here the bottom line is the south pole $\mathbf{s} = (0,0,-1)$, the top line is the north pole $\mathbf{n} = (0,0,+1)$ and other horizontal lines are circles contained in a plane of the form $z = z_0$; vertical lines are meridians, i.e., half circles joining the two poles. Each small sphere is drawn here as a photo of a transparent sphere: a curve in the front is drawn a bit thicker. The vector perpendicular to the page pointing towards the reader is $e_1 + e_3$ with $e_3 - e_1$ pointing up and e_2 to the right of the reader. The base point e_1 is drawn as a thick dot.

Consider the equator of the unit sphere $\mathbb{S}^2$ contained in the plane $z = 0$. Fix a unit normal vector $N = (0,0,1)$ to the plane and call it *up*. Consider six equally spaced points $P_0 = P_3 = (1/2, -\sqrt{3}/2, 0)$, $Q_0 = Q_3 = (1,0,0)$, $P_1 = (1/2, \sqrt{3}/2, 0)$, $Q_1 = -P_0$, $P_2 = -Q_0$ and $Q_2 = -P_1$ along the equator. Notice that $Q_i/2$ is the midpoint of the segment P_iP_{i+1}. For $\alpha \in \mathbb{R}$, let $\tilde{\alpha} = \arcsin(\sin(\alpha)/2)$ so that $\tilde{\alpha} \in [-\pi/6, \pi/6]$. Let

$$Q_i^{\pm}(\alpha) = \cos(\alpha - \tilde{\alpha})Q_i \pm \sin(\alpha - \tilde{\alpha})N,$$

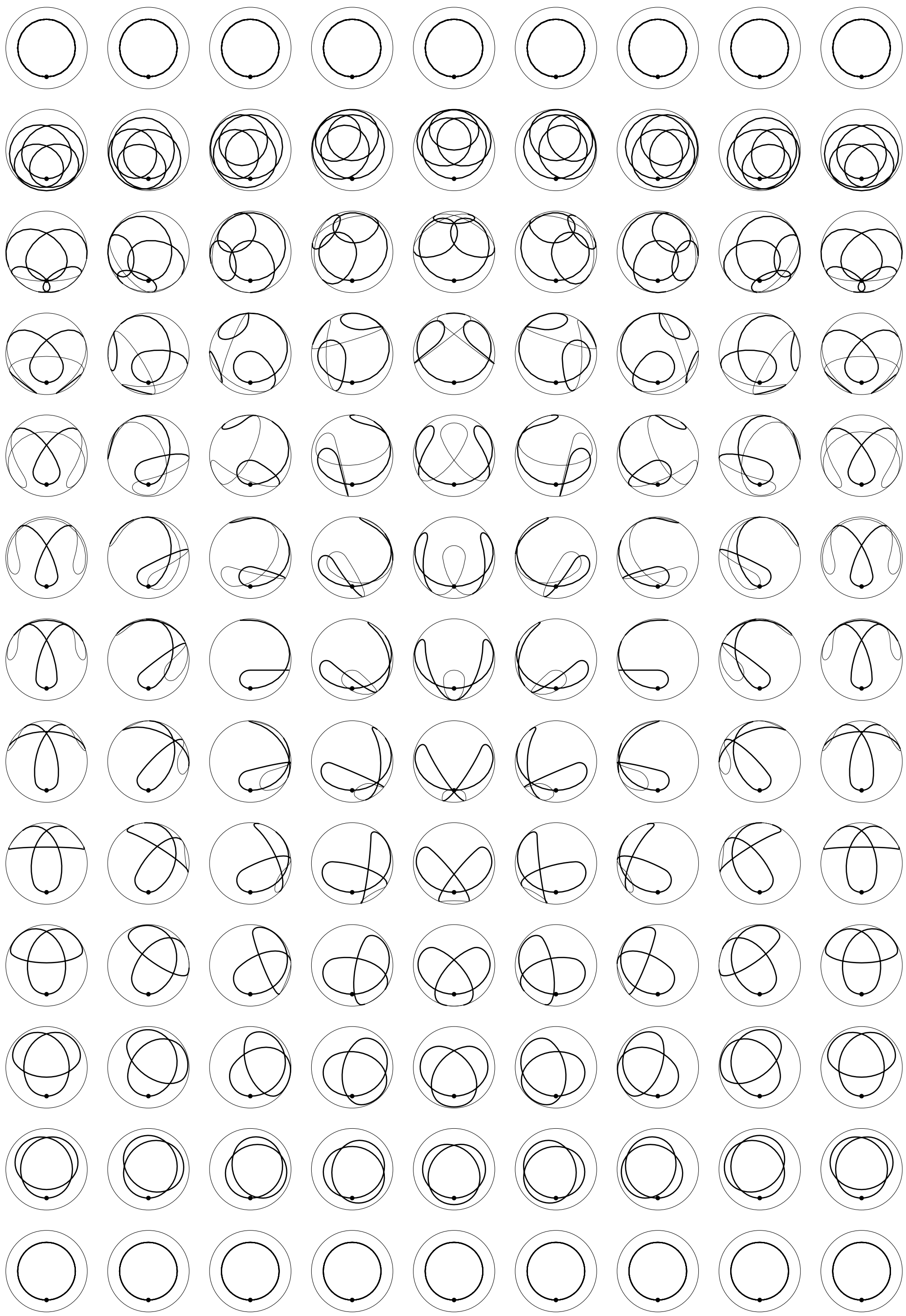

Figure 6: A sketch of $g_0 : \mathbb{S}^2 \to \mathcal{L}_{\mathbf{1}}$.

so that $Q_i^+(0) = Q_i^-(0) = Q_i$, $Q_i^-(\alpha) = Q_i^+(-\alpha)$ and Q_i^+ (a function of α) parametrizes the circle passing through $\pm Q_i$ and $\pm N$. Equivalently, $Q_i^+(\alpha)$ is the only point on the unit sphere such that the vector $Q_i^+(\alpha) - (Q_i/2)$ is a positive multiple of $(\cos\alpha)Q_i + (\sin\alpha)N$. Let $\mathcal{C}_i^\pm(\alpha)$ be the circle containing P_i, $Q_i^\pm(\alpha)$ and P_{i+1}, so that $\mathcal{C}_i^\pm(\alpha)$ has radius $\cos(\tilde\alpha)$ and is contained in a plane perpendicular to $-\cos(\alpha)N \pm \sin(\alpha)Q_i$. Let $A_i^\pm(\alpha) \subset \mathcal{C}_i^\pm(\alpha)$ be the arc from P_i through $Q_i^\pm(\alpha)$ to P_{i+1}.

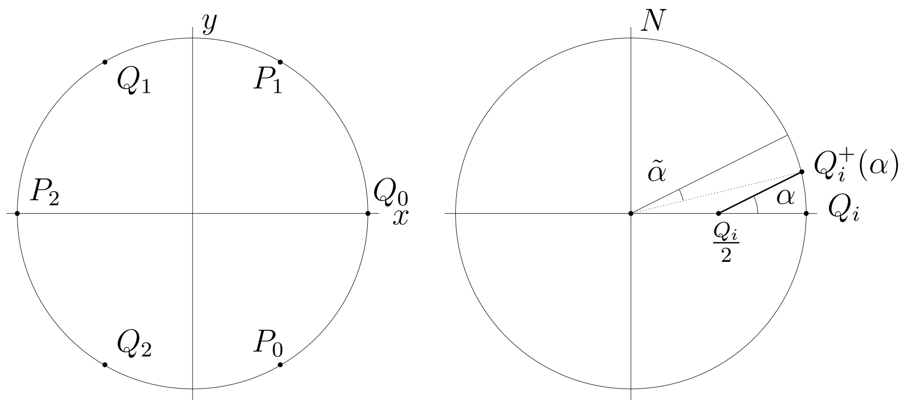

Figure 7: Constructing P_i, Q_i and $Q_i^+(\alpha)$.

Orient the circles $\mathcal{C}_i^\pm(\alpha)$ and the arcs $A_i^\pm(\alpha)$ from P_i through $Q_i^\pm(\alpha)$ to P_{i+1}. The circle $\mathcal{C}_i^\pm(\alpha)$ has geodesic curvature equal to $\mp\tan(\tilde\alpha)$. Parametrize the arcs $A_0^+(\alpha)$, $A_1^-(\alpha)$, $A_2^+(\alpha)$, $A_0^-(\alpha)$, $A_1^+(\alpha)$, $A_2^-(\alpha)$ and $A_0^+(\alpha)$ by a multiple of arc length using the domains $[-1/12, 1/12]$, $[1/12, 3/12]$, $[3/12, 5/12]$, $[5/12, 7/12]$, $[7/12, 9/12]$, $[9/12, 11/12]$ and $[11/12, 13/12]$, respectively. Concatenate the above parametrizations to define a parametrization $\beta_\alpha : [0,1] \to \mathbb{S}^2$ by a multiple of arc length of a curve $C_\alpha \subset \mathbb{S}^2$. In particular, $\beta_\alpha(0) = Q_0^+(\alpha)$, $\beta_\alpha(1/12) = P_1$, $\beta_\alpha(2/12) = Q_1^-(\alpha)$ and so on. Notice that β_α is of class C^1, even at the points $t = j/12$. The curve β_α is an immersion; its geodesic curvature is always in the interval $[-\sqrt{3}/3, +\sqrt{3}/3]$, with the extremes assumed for $\alpha = \pi/2 + k\pi$, $k \in \mathbb{Z}$. The curve β_0 is the equator covered twice; β_π is the equator covered four times with the opposite orientation. Define $B_\alpha = \mathfrak{F}_{\beta_\alpha} : [0,1] \to SO_3$ as usual; lift this to define $\tilde B_\alpha : [0,1] \to \mathbb{S}^3$ with $\tilde B_0(0) = \mathbf{1}$ and $\tilde B_\alpha(t)$ a continuous function of α and t, 1-periodic in t and 4π-periodic in α.

Define $\mathbf{h} = \exp(\pi\mathbf{j}/8)$; notice that

$$\mathbf{h}\mathbf{i}\mathbf{h}^{-1} = \hat{\mathbf{i}} = \frac{\mathbf{i}-\mathbf{k}}{\sqrt{2}}, \quad \mathbf{h}\mathbf{k}\mathbf{h}^{-1} = \hat{\mathbf{k}} = \frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}}.$$

Define $\tilde\Gamma_\alpha(t) = \tilde B_\alpha(t)\mathbf{h}^{-1}$, $\Gamma_\alpha = \Pi \circ \tilde\Gamma_\alpha$ and (of course)

$$\gamma_\alpha(t) = \Gamma_\alpha(t)e_1 = B_\alpha(t)\frac{e_1+e_3}{\sqrt{2}} = \frac{\sqrt{2}}{2}\beta_\alpha(t) + \frac{\sqrt{2}}{2}\mathbf{n}_{\beta_\alpha}(t).$$

Geometrically, γ_α is the concatenation of six circle arcs $\tilde{A}_i^\pm$, obtained from $A_i^\pm$ by increasing or decreasing the radius by $\pi/4$. The geodesic curvature of γ_α is

$$\kappa_{\gamma_\alpha}(t) = \begin{cases} \tan\left(\frac{\pi}{4} - \tilde{\alpha}\right), & t \in [0, \frac{1}{12}) \cup (\frac{3}{12}, \frac{5}{12}) \cup (\frac{7}{12}, \frac{9}{12}) \cup (\frac{11}{12}, 1], \\ \tan\left(\frac{\pi}{4} + \tilde{\alpha}\right), & t \in (\frac{1}{12}, \frac{3}{12}) \cup (\frac{5}{12}, \frac{7}{12}) \cup (\frac{9}{12}, \frac{11}{12}). \end{cases}$$

and therefore always in the interval $[2-\sqrt{3}, 2+\sqrt{3}]$, with extreme values assumed for $\alpha = \pi/2$. Furthermore, $\operatorname{tot}(\gamma_\alpha)$ is a strictly increasing function of $\alpha \in [0, \pi]$ with $\operatorname{tot}(\gamma_0) = 4\pi$, $\operatorname{tot}(\gamma_\pi) = 8\pi$. Notice that

$$\tilde{\Gamma}_\alpha\left(t + \frac{1}{3}\right) = \exp\left(\frac{4\pi}{3}\mathbf{k}\right)\tilde{\Gamma}_\alpha(t); \quad \tilde{\Gamma}_\alpha\left(t + \frac{1}{2}\right) = -\tilde{\Gamma}_{-\alpha}(t).$$

Define $g_0 : \mathbb{S}^2 \to \mathcal{L}_\mathbf{1}$ by

$$g_0(p)(t) = \left(\Gamma_\alpha\left(\frac{\theta}{6\pi}\right)\right)^{-1} \gamma_\alpha\left(t + \frac{\theta}{6\pi}\right), \quad p = (\cos\theta\sin\alpha, \sin\theta\sin\alpha, -\cos\alpha).$$

Notice that g_0 is well defined.

This construction yields the following result, due to Little ([15]); we state and prove it here in order to get used to the above construction which shall be needed later.

Lemma 5.1 *Let n be a positive integer, $n > 1$. The curves ν_n and ν_{n+2} are in the same connected component of $\mathcal{L}_I$.*

Proof: The case $n = 2$ follows from the above construction of g_0. For larger n, just consider ν_n as a concatenation of ν_{n-2} with ν_2, keep ν_{n-2} fixed and apply the above construction to move from ν_2 to ν_4, thus obtaining a path in $\mathcal{L}_I$ from ν_n to ν_{n+2}. ■

We prove an auxiliary result concerning the curves β_α and γ_α for later use. A *common oriented tangent* to two oriented circles is an oriented geodesic which is tangent to both circles, with compatible orientation at tangency points. Thus, for instance, $\mathcal{C}_i^\pm(\alpha)$ and $\mathcal{C}_{i+1}^\mp(\alpha)$ are tangent at P_{i+1}; they also have a common oriented tangent, a geodesic passing through P_{i+1}. Let $\mathcal{C}_\mathbf{k}$ (resp. $\mathcal{C}_{\hat{\mathbf{k}}}$) be the great circle and subgroup of $\mathbb{S}^3$ of points of the form $\exp(s\mathbf{k})$ (resp. $\exp(s\hat{\mathbf{k}})$), $s \in \mathbb{R}$. Recall that we write $\tilde{B}_\alpha(t_0; t_1) = (\tilde{B}_\alpha(t_0))^{-1}\tilde{B}_\alpha(t_1)$.

Lemma 5.2 *Let $\alpha \in (0, \pi)$.*

(a) *The common oriented tangents between two distinct circles among $\mathcal{C}_i^\pm(\alpha)$, $i = 0, 1, 2$, are the geodesics passing through the tangency point P_{i+1} between $\mathcal{C}_i^\pm(\alpha)$ and $\mathcal{C}_{i+1}^\mp(\alpha)$, and only these.*

(b) For $t_0, t_1 \in [0,1)$, if $\tilde{B}_\alpha(t_0; t_1) \in \mathcal{C}_{\mathbf{k}}$ then $t_0 = t_1$.

(c) For $t_0, t_1 \in [0,1)$, if $\tilde{\Gamma}_\alpha(t_0; t_1) \in \mathcal{C}_{\hat{\mathbf{k}}}$ then $t_0 = t_1$.

Proof: Recall that our circles $\mathcal{C}_i^{\pm}(\alpha)$ are not geodesics. Each circle therefore defines a disk (the smaller connected component of the complement).

For item (a), we have three essentially different pairs of circles to consider: $(\mathcal{C}_0^+(\alpha), \mathcal{C}_0^-(\alpha))$, $(\mathcal{C}_0^+(\alpha), \mathcal{C}_1^+(\alpha))$ and $(\mathcal{C}_0^+(\alpha), \mathcal{C}_1^-(\alpha))$. In the first case, the two circles have opposite orientations and therefore the circles and corresponding disks would have to lie on opposite sides of a common oriented tangent. Since the open disks intersect, no common oriented tangent exists. In the second case orientations agree and therefore both disks would lie on the same side of a common oriented tangent. But the union of the two closed disks contains the arc from P_0 through P_1 to P_2 and therefore is not contained in a hemisphere. Thus also in this case no common oriented tangent exists. Finally, in the third case again the two circles have opposite orientations and therefore the disks must lie on opposite sides of a common oriented tangent. But the closed disks touch at P_1: the common oriented tangent must therefore pass through this point, completing the proof of the first claim.

For item (b), assume by contradiction that $\tilde{B}_\alpha(t_0; t_1) = \exp(s\mathbf{k})$, $t_0 \neq t_1$. Consider the curve $\tilde{\Gamma} : [0,1] \to \mathbb{S}^3$ given by $\tilde{\Gamma}(t) = \tilde{B}_\alpha(t_0) \exp(2\pi t \mathbf{k})$. Notice that $\tilde{\Gamma}(0) = \tilde{B}_\alpha(t_0)$ and $\tilde{\Gamma}(s/(2\pi)) = \tilde{B}_\alpha(t_1)$. Also $\tilde{\Gamma}(t) = \tilde{\mathfrak{F}}_\gamma(t)$ for γ the geodesic $\gamma(t) = B_\alpha(t_0)(\cos(4\pi t), \sin(4\pi t), 0)$. Thus γ is an oriented tangent to the curve C_α at two distinct points. These two points must belong to different circles $\mathcal{C}_i^{\pm}$ for a circle can not be twice tangent to the same geodesic. But this contradicts item (a).

Finally, for item (c), notice that

$$\tilde{\Gamma}_\alpha(t_0; t_1) = \mathbf{h} \tilde{B}_\alpha(t_0; t_1) \mathbf{h}^{-1}$$

and therefore $\tilde{\Gamma}_\alpha(t_0; t_1) = \exp(s\hat{\mathbf{k}})$ implies

$$\tilde{B}_\alpha(t_0; t_1) = \mathbf{h}^{-1} \exp(s\hat{\mathbf{k}}) \mathbf{h} = \exp\left(s\mathbf{h}^{-1}\hat{\mathbf{k}}\mathbf{h}\right) = \exp(s\mathbf{k}).$$

Thus item (b) implies item (c). ■

Proposition 5.3 *The map g_0 is a generator of $\pi_2(\mathcal{I}_{\mathbf{1}})$.*

Proof: Given $g : \mathbb{S}^2 \to \mathcal{I}_{\mathbf{1}}$, define $\hat{g} : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$ by $\hat{g}(p,t) = \tilde{\mathfrak{F}}_{g(p)}(t)$, where we identify $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Let $N(g)$ be the degree of $\hat{g}$. Clearly $N(g)$ is invariant by homotopy and $N(g_1 * g_2) = N(g_1) + N(g_2)$ (where $*$ is the operation defining π_2). We prove that $|N(g_0)| = 1$, completing the proof of the proposition.

From Lemma 5.2 above, $\tilde{\mathfrak{F}}_{g(p)}(t) = \mathbf{1}$ implies either $t = 0$ or $p = \mathbf{n}$ (the north pole) and $t = 1/2$. For the purpose of computing the degree of $\hat{g}_0$, we deform it to define another map $h : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$ corresponding to closed curves coinciding with $g_0(p)$ except at a small neighborhood of $t = 0$, where they cross the xz plane at a point $(\cos(\epsilon), 0, \sin(\epsilon))$, $\epsilon > 0$. Thus the only preimage of $\mathbf{1}$ under h is $(\mathbf{n}, 1/2)$ and we are left with verifying that it is topologically transversal.

Alternatively, again from Lemma 5.2 above, the preimages under $\hat{g}_0$ of $-\mathbf{1}$ are exactly $(\mathbf{s}, 1/2)$, $(\mathbf{n}, 1/4)$ and $(\mathbf{n}, 3/4)$. We are left with verifying that all three are topologically transversal and that the sign of the first is different from the sign of the last two, again implying $|N(g_0)| = 1$.

Unfortunately, g_0 is not differentiable at $\mathbf{s}$ but it does admit directional derivatives: that is enough. We go back to the construction of $\tilde{B}_\alpha$ and $\tilde{\Gamma}_\alpha$ in order to compute directional derivatives of g_0. For $t \in [-1/12, 1/12]$, we may translate the geometric description above as:

$$\tilde{B}_\alpha(t) = \exp\left(\frac{\alpha}{2}\mathbf{j}\right) \exp\left(u(\alpha)t\mathbf{k}\right) \exp\left(v(\alpha)\mathbf{j}\right),$$
$$u(\alpha) = 6 \arccos\left(\frac{\cos\alpha}{\sqrt{4 - \sin^2\alpha}}\right), \quad v(\alpha) = -\frac{1}{2}\arcsin\left(\frac{1}{2}\sin\alpha\right).$$

Notice that $u'(0) = 0$, $v'(0) = -1/4$. Let $\tilde{B}^\bullet_\alpha(t)$ be the derivative of $\tilde{B}_\alpha(t)$ with respect to α; we have $(\tilde{B}_0(t))^{-1}\tilde{B}^\bullet_0(t) = w(t)$ where the auxiliary function w is defined by

$$w(t) = \frac{1}{4}\left((2\cos(4\pi t) - 1)\,\mathbf{j} + (-2\sin(4\pi t))\,\mathbf{i}\right).$$

It is now easy to obtain similar formulas for other intervals and to deduce that

$$(\tilde{B}_0(t))^{-1}\tilde{B}^\bullet_0(t) = \begin{cases} w(t - t_0), & t \in \left[t_0 - \frac{1}{12}, t_0 + \frac{1}{12}\right], t_0 = \frac{k}{3}, \\ -w(t - t_0), & t \in \left[t_0 - \frac{1}{12}, t_0 + \frac{1}{12}\right], t_0 = \frac{k}{3} + \frac{1}{6}, \end{cases} \qquad k \in \mathbb{Z}.$$

Thus $(\tilde{B}_0(t))^{-1}\tilde{B}^\bullet_0(t)$, as a function of t, performs three full turns around the origin in the plane spanned by $\mathbf{i}$ and $\mathbf{j}$. We now have that $(B_0(t; t + \frac{1}{2}))^{-1}B^\bullet_0(t; t + \frac{1}{2})$ performs one full turn around the origin when t goes from 0 to $1/3$.

Recall that $\tilde{\Gamma}_\alpha(t) = \tilde{B}_\alpha(t)\mathbf{h}^{-1}$ and therefore $\Gamma_\alpha(t; t+\frac{1}{2}) = \mathbf{h}B_\alpha(t; t+\frac{1}{2})\mathbf{h}^{-1}$ and we have that $(\Gamma_0(t; t + \frac{1}{2}))^{-1}\Gamma^\bullet_0(t; t + \frac{1}{2})$ performs one full turn around the origin when t goes from 0 to $1/3$, but now in the plane spanned by $\hat{\mathbf{i}}$ and $\mathbf{j}$. A similar computation shows that, when t goes from 0 to $1/3$, $(\Gamma_\pi(t; t + \frac{1}{2}))^{-1}\Gamma^\bullet_\pi(t; t + \frac{1}{2})$ also performs one full turn around the origin in the same plane.

Translating this back to g_0 shows that when p describes a small circle around either the south or the north pole, $g_0(p)(1/2)$ describes a small simple closed curve around $(g_0(\mathbf{s}))(1/2) = (g_0(\mathbf{n}))(1/2) = e_1$, with $(g_0(p))'(1/2) \approx (g_0(\mathbf{s}))'(1/2) =$

$(g_0(\mathbf{n}))'(1/2) = e_2$. The reader should check in Figure 6 that this is indeed the case: said simple closed curve is drawn clockwise when we go left to right along either the second row from the bottom (around $\mathbf{s}$) or the second from the top (around $\mathbf{n}$). It follows from Lemma 5.2 (and can be checked in Figure 6) that the same holds for other values of t. Thus, for instance, when p describes a left to right small circle around the north pole, $g_0(p)(1/4)$ and $g_0(p)(3/4)$ both describe simple closed curves around e_1, also oriented clockwise.

Finally, translating these results to $\hat{g}_0$, the image under $\hat{g}_0$ of a small sphere around $(\mathbf{n}, 1/2)$ wraps once around $\mathbf{1} = \hat{g}_0((\mathbf{n}, 1/2))$, proving topological transversality at this point. Similarly, the image of a small sphere around $(\mathbf{s}, 1/2)$, $(\mathbf{n}, 1/4)$ or $(\mathbf{n}, 3/4)$ wraps once around $-\mathbf{1}$; the orientation is different for the first point because, from the point of view of $\mathbb{S}^2$, a left to right circle near $\mathbf{n}$ and a left to right circle near $\mathbf{s}$ have opposive orientations. ■

6 Adding loops

In this section we present a few facts related to adding loops to a curve (or family of curves), including the proof of Proposition 1.3. This is of course similar to the proof of the Hirsch-Smale theorem ([12], [26]). The reader familiar with Gromov's ideas will also recognize this as an easy instance of the h-principle ([7], [11]); others will be reminded of Thurston's method for performing the sphere eversion by corrugations ([14]).

We need a precise version for the notion of adding n loops at a point t_0 of a curve γ as in Figure 8. For $\gamma \in \mathcal{I}$, $t_0 \in (0,1)$ and n a positive integer let $\gamma^{[t_0 \# n]} \in \mathcal{I}$ be defined by

$$\gamma^{[t_0\#n]}(t) = \begin{cases} \gamma(t), & 0 \le t \le t_0 - 2\epsilon; \\ \gamma(2t - t_0 + 2\epsilon), & t_0 - 2\epsilon \le t \le t_0 - \epsilon; \\ \mathfrak{F}_\gamma(t_0)\nu_n\left(\frac{t - t_0 + \epsilon}{2\epsilon}\right), & t_0 - \epsilon \le t \le t_0 + \epsilon; \\ \gamma(2t - t_0 - 2\epsilon), & t_0 + \epsilon \le t \le t_0 + 2\epsilon; \\ \gamma(t), & t_0 + 2\epsilon \le t \le 1; \end{cases}$$

here $\epsilon > 0$ is taken sufficiently small. In words, we follow γ normally almost until t_0: we then run a little in order to have room to insert ν_n (appropriately moved to the correct position), run a little again and then continue as γ. Since reparametrizations of curves are not particularly interesting (the group of orientation-preserving diffeomorphisms of $[0,1]$ is contractible) the precise value of ϵ is not particularly interesting either.

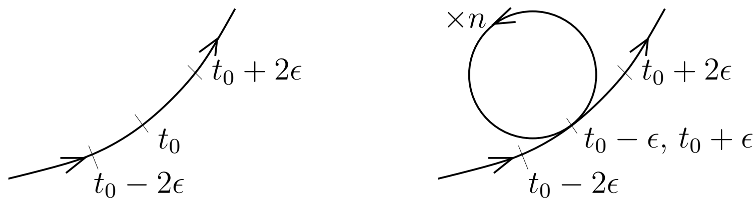

Figure 8: Curves γ and $\gamma^{[t_0\#n]}$.

For $t_0 = 0$ and $t_0 = 1$, the definition must be slightly different so that endpoints remain untouched. Thus, for instance,

$$\gamma^{[0\#n]}(t) = \begin{cases} \nu_n\left(\frac{t}{\epsilon}\right), & 0 \le t \le \epsilon; \\ \gamma(2t - 2\epsilon), & \epsilon \le t \le 2\epsilon; \\ \gamma(t), & 2\epsilon \le t \le 1. \end{cases}$$

Notice that if $\gamma \in \mathcal{I}_z$ then $\gamma^{[t_0\#n]} \in \mathcal{I}_{(-\mathbf{1})^n z}$. Also, if γ is locally convex then so is $\gamma^{[t_0\#n]}$, with $\operatorname{tot}(\gamma^{[t_0\#n]}) = 2\pi n + \operatorname{tot}(\gamma)$. We use the notation $\gamma^{[t_0\#n_0;t_1\#n_1]}$ for $(\gamma^{[t_0\#n_0]})^{[t_1\#n_1]} = (\gamma^{[t_1\#n_1]})^{[t_0\#n_0]}$. Also, given $t_0 : K \to (0,1)$ and $f : K \to \mathcal{I}_Q$ continuous functions, let $f^{[t_0\#n]} : K \to \mathcal{I}_Q$ be defined by $f^{[t_0\#n]}(p) = (f(p))^{[t_0(p)\#n]}$.

Notice that in $\mathcal{I}_Q$ it is easy to introduce a pair of loops at any point of the curve.

Lemma 6.1 *Let K be a compact set, $Q \in SO_3$ and n a positive even integer. Let $t_0 : K \to (0,1)$ and $f : K \to \mathcal{I}_Q$ be continuous functions. Then f and $f^{[t_0\#n]}$ are homotopic.*

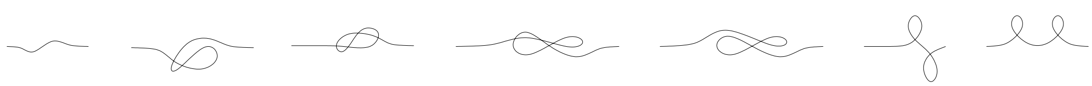

Figure 9: How to add two small loops to a curve in $\mathcal{I}_Q$.

Proof: The process is illustrated in Figure 9; in the final step one of the loops becomes big, goes around the sphere and shrinks again. We do not think that an explicit formula is helpful. ■

Somewhat similarly, for locally convex curves we have the following.

Lemma 6.2 *Let K be a compact set, $Q \in SO_3$ and $n > 1$ an integer. Let $t_0 : K \to (0,1)$ and $f : K \to \mathcal{L}_Q$ be continuous functions. Then $f^{[t_0\#n]}$ and $f^{[t_0\#(n+2)]}$ are homotopic, i.e., there exists $H : [0,1] \times K \to \mathcal{L}_Q$ with $H(0,p) = f^{[t_0\#n]}(p)$, $H(1,p) = f^{[t_0\#(n+2)]}(p)$. We may furthermore assume that $2\pi n + \operatorname{tot}(f(p)) \le \operatorname{tot}(H(s,p)) \le 2\pi(n+2) + \operatorname{tot}(f(p))$.*

Proof: We use the path from ν_n to ν_{n+2} in $\mathcal{L}_I$ constructed in Lemma 5.1 to modify the curve $f(p)$ in the interval $[t_0(p)-\epsilon, t_0(p)+\epsilon]$ only. ■

The above statement actually holds for $n = 1$ (by borrowing a little elbow room from the surrounding curve) but we do not need this observation. On the other hand, the statement critically fails for $n = 0$. Indeed, we know (from Fenchel, Little and others; [8], [15]) that ν_1 and ν_3 are not in the same connected component of $\mathcal{L}_I$.

We now need a construction corresponding to spreading loops along the curve, as in Figure 2. For $\gamma \in \mathcal{I}$ and $n > 0$, define

$$\gamma^{[\#(2n)]} = \gamma^{[t_0\#1;t_1\#2;t_2\#2;\cdots;t_{n-1}\#2;t_n\#1]}; \quad t_j = \frac{j}{n}.$$

We assume here that the same $\epsilon > 0$ is used for each loop so that

$$t \in [0,1], |t - t_j| \le \epsilon \quad \Rightarrow \quad \gamma^{[\#(2n)]}(t) = \mathfrak{F}_\gamma(t_j)\nu_1\left(\frac{t-t_j}{\epsilon}\right).$$

We now have $2n$ loops attached to the curve γ. We need another construction, however, to smooth out the remaining small arcs of γ, in order to define a curve $\gamma^{[\flat(2n)]} : [0,1] \to \mathbb{S}^2$ which, for sufficiently large n, will be similar to $\gamma^{[\#(2n)]}$ and locally convex (even when the original curve γ is not). For $1 \le j \le n$, let

$$t_{j,0} = t_{j-1} + \frac{7\epsilon}{8}, \quad t_{j,\frac{1}{2}} = \frac{t_{j-1}+t_j}{2} = \frac{2j-1}{2n}, \quad t_{j,1} = t_j - \frac{7\epsilon}{8}$$

and define

$$\gamma^{[\flat(2n)]}(t) = \gamma^{[\#(2n)]}(t), \qquad t \notin \bigcup_{0<j\le n} (t_{j,0}, t_{j,1}).$$

It remains to define the arcs $\gamma^{[\flat(2n)]} : [t_{j,0}, t_{j,1}] \to \mathbb{S}^2$. Notice that $\mathfrak{F}_\gamma(t_{j,\frac{1}{2}}) = \mathfrak{F}_{\gamma^{[\#(2n)]}}(t_{j,\frac{1}{2}})$ and, for large n,

$$\mathfrak{F}_{\gamma^{[\#(2n)]}}(t_{j,0}) \approx \mathfrak{F}_\gamma(t_{j,\frac{1}{2}})\mathfrak{F}_{\nu_1}(-1/8), \quad \mathfrak{F}_{\gamma^{[\#(2n)]}}(t_{j,1}) \approx \mathfrak{F}_\gamma(t_{j,\frac{1}{2}})\mathfrak{F}_{\nu_1}(+1/8);$$

We may therefore apply Lemma 3.1 to conclude that there exists a unique arc of ellipse parametrized by projective arc-length $\gamma^{[\flat(2n)]} : [t_{j,0}, t_{j,1}] \to \mathbb{S}^2$ with

$$\mathfrak{F}_{\gamma^{[\flat(2n)]}}(t_{j,0}) = \mathfrak{F}_{\gamma^{[\#(2n)]}}(t_{j,0}), \qquad \mathfrak{F}_{\gamma^{[\flat(2n)]}}(t_{j,1}) = \mathfrak{F}_{\gamma^{[\#(2n)]}}(t_{j,1}),$$
$$\gamma^{[\flat(2n)]}(t_{j,\frac{1}{2}}) = \gamma^{[\#(2n)]}(t_{j,\frac{1}{2}}).$$

This completes the definition of $\gamma^{[\flat(2n)]}$. For clarity, we rephrase it more informally. Draw circles tangent to the curve at the points $t_j = j/n$, one for $j = 0$ or $j = n$, two for other values of j. In order to jump from one circle to the next, draw an arc of ellipse.

Let $f : K \to \mathcal{I}_z$ be a continuous function. Define $f^{[\flat(2n)]} : K \to \mathcal{I}_z$ by $f^{[\flat(2n)]}(p) = (f(p))^{[\flat(2n)]}$. Given f, for sufficiently large n the function $f^{[\flat(2n)]}$ is well defined and continuous and its image is contained in $\mathcal{L}_z$.

Lemma 6.3 *Let K be a compact set, $f : K \to \mathcal{I}_z$ and $t_0 : K \to (0,1)$ continuous maps. Then, for sufficiently large n, the following properties hold.*

(a) The image of $f^{[\flat(2n)]}$ is contained in $\mathcal{L}_z$.

(b) The function $f^{[\flat(2n)]}$ is homotopic to f, i.e., there exists $H_b : [0,1] \times K \to \mathcal{I}_z$ such that $H_b(0,\cdot) = f^{[\flat(2n)]}$ and $H_b(1,\cdot) = f$.

(c) The function $f^{[\flat(2n)]}$ is homotopic to $f^{[t_0\#(2n)]}$, i.e., there exists $H_c : [0,1] \times K \to \mathcal{I}_z$ such that $H_c(0,\cdot) = f^{[\flat(2n)]}$ and $H_c(1,\cdot) = f^{[t_0\#(2n)]}$.

(d) If the image of f is contained in $\mathcal{L}_z$ then the image of the homotopy H_c is also contained in $\mathcal{L}_z$.

Notice that even if the image of f is contained in $\mathcal{L}_z$ we do not claim that f is homotopic to $f^{[\flat(2n)]}$ in $\mathcal{L}_z$; on the contrary, we shall soon see that this is not always the case. Notice also that Proposition 1.3 follows directly from this lemma.

Proof: Item (a) follows from the remarks above.

For item (c), notice first that the functions $f^{[t_0\#(2n)]}$ and $f^{[\#(2n)]}$ are homotopic: the homotopy consists of merely rolling loops along the curve. More precisely, for $\tilde{t}_j(s) = sj/n + (1-s)t_0$, define

$$H_1(s,p) = (f(p))^{[\tilde{t}_0(s)\#1;\tilde{t}_1(s)\#2;\cdots;\tilde{t}_{n-1}(s)\#2;\tilde{t}_n(s)\#1]}.$$

We next verify that, for sufficiently large n, the functions $f^{[\#(2n)]}$ and $f^{[\flat(2n)]}$ are homotopic. Let $Q_j(p) = (\mathfrak{F}_{f(p)}(t_{j,\frac{1}{2}}))^{-1} \in SO_3$, where $t_{j,0}, t_{j,\frac{1}{2}}, t_{j,1}$ are as in the construction of $f^{[\flat(2n)]}$. We have

$$Q_j(p)\mathfrak{F}_{(f(p))^{[\flat(2n)]}}(t_{j,0}) = Q_j(p)\mathfrak{F}_{(f(p))^{[\#(2n)]}}(t_{j,0}) \approx \Pi(\exp(-\pi\hat{\mathbf{k}}/8)),$$
$$Q_j(p)\mathfrak{F}_{(f(p))^{[\flat(2n)]}}(t_{j,1}) = Q_j(p)\mathfrak{F}_{(f(p))^{[\#(2n)]}}(t_{j,1}) \approx \Pi(\exp(+\pi\hat{\mathbf{k}}/8)).$$

Thus, for sufficiently large n, the arcs

$$Q_j(p)(f(p))^{[\flat(2n)]}, Q_j(p)(f(p))^{[\#(2n)]} : [t_{j,0}, t_{j,1}] \to \mathbb{S}^2$$

are *graphs*, in the sense that the first coordinate $x : [t_{j,0}, t_{j,1}] \to [x_-, x_+]$ is an increasing diffeomorphism (with $x_\pm \approx \pm 1/2$) and y and z can be considered functions of x. Since the space of increasing diffeomorphisms of an interval is contractible, we may construct a homotopy from $f^{[\#(2n)]}$ to a suitable reparametrization f_1 of $f^{[\#(2n)]}$ in each $[t_{j,0}, t_{j,1}]$ for which the function x above is the same as for $f^{[\flat(2n)]}$. We may then join f_1 and $f^{[\flat(2n)]}$ by performing a convex combination followed by projection to $\mathbb{S}^2$, completing the proof of (c).

For item (d), we observe that if the curves are locally convex then both constructions above remain in the space of locally convex curves.

Finally, for item (b), we know from Lemma 6.1 that f is homotopic to $f^{[t_0\#(2n)]}$ and from item (c) that $f^{[t_0\#(2n)]}$ is homotopic to $f^{[\flat(2n)]}$. ■

As we shall see later, a function $f : K \to \mathcal{L}_z \subset \mathcal{I}_z$ may be homotopic to a constant in $\mathcal{I}_z$ but not in $\mathcal{L}_z$. The following proposition shows that this changes if we add loops.

Proposition 6.4 *Let n be an even positive integer. Let K be a compact set and let $f : K \to \mathcal{L}_z \subset \mathcal{I}_z$ a continuous function. Then f is homotopic to a constant in $\mathcal{I}_z$ if and only if $f^{[t_0\#n]}$ is homotopic to a constant in $\mathcal{L}_z$.*

Proof: In $\mathcal{I}_z$, f and $f^{[t_0\#n]}$ are homotopic, proving one implication. For the other implication, let $H : K \times [0,1] \to \mathcal{I}_z$ be a homotopy with $H(\cdot,0) = f$, $H(\cdot,1)$ constant. By Lemma 6.3, for sufficiently large even m, the image of $H^{[\flat(2m)]}$ is contained in $\mathcal{L}_z$. This implies that $f^{[\flat(2m)]}$ is homotopic in $\mathcal{L}_z$ to a constant. From Lemma 6.3, $f^{[t_0\#(2m)]}$ is homotopic to $f^{[\flat(2m)]}$ in $\mathcal{L}_z$ and therefore the proposition is proved for large even n. The general case now follows from Lemma 6.2. ■

A map $f : K \to \mathcal{L}_Q$ is *loose* if f is homotopic to $f^{[t_0\#2]}$ (in $\mathcal{L}_Q$) and *tight* otherwise. Lemma 6.2 shows that $f^{[t_0\#2]}$ is loose. If $K = \{p_0\}$ consists of a single point then a function $f : K \to \mathcal{L}_Q$ is essentially a curve $\gamma_0 = f(p_0)$; f is then tight if and only if γ_0 is convex (for $Q = I$ this follows from the results of Little; otherwise from Anisov, Shapiro and Shapiro; [15], [1], [24], [25]). As we shall see, the map g_0 constructed above is tight: this observation will be crucial.

From now on we consider that a main question is, given $f : K \to \mathcal{L}_Q$, to decide whether f is loose or tight. We shall see a few key examples of tight maps and we shall prove that large classes of maps are loose. It is sometimes important to have estimates of the total curvature during the homotopy.

Lemma 6.5 *Let $f_0, f_1 : K \to \mathcal{L}_Q$ be homotopic with f_0 loose. Then f_1 is loose.*

Proof: Let $t_0 : K \to (0,1)$ be a continuous function and let $H : [0,1] \times K \to \mathcal{L}_Q$ be a homotopy from f_0 to f_1. Let $H^{[t_0\#2]}$ be defined by $H^{[t_0\#2]}(s,p) = (H(s,p))^{[t_0\#2]}$; clearly, this is a homotopy from $f_0^{[t_0\#2]}$ to $f_1^{[t_0\#2]}$. Thus, if f_0 is homotopic to $f_0^{[t_0\#2]}$ then f_1 is homotopic to $f_1^{[t_0\#2]}$, as desired. ■

We finish this section with a more complicated lemma which allows us to see that many maps $f : K \to \mathcal{L}_Q$ are loose.

Lemma 6.6 *Let $Q \in SO_3$. Let K be a compact manifold and $f : K \to \mathcal{L}_Q$ a continuous map. Assume that:*

- *$t_0 \in (0,1)$ and $t_1, t_2, \ldots, t_J : K \to (0,1)$ are continuous functions with $t_0 < t_1 < t_2 < \cdots < t_J$;*
- *$K = \bigcup_{1 \le j \le J} U_j$, where $U_j \subset K$ are open sets;*
- *there exist continuous functions $g_j : U_j \to \mathcal{L}_Q$ such that, for all $p \in U_j$, we have $f(p) = (g_j(p))^{[t_j(p)\#2]}$.*

Then f is loose, i.e., there exists $H : [0,1] \times K \to \mathcal{L}_Q$ with $H(0,p) = f(p)$, $H(1,p) = (f(p))^{[t_0\#2]}$. We may furthermore assume that

$$\operatorname{tot}(f(p)) \leq \operatorname{tot}(H(s,p)) \leq 4\pi + \operatorname{tot}(f(p)).$$

Proof: Our proof proceeds by induction on J. For $J = 1$ we have $U_1 = K$ and therefore $f = g_1^{[t_1\#2]}$, which is known from Lemma 6.2 to be loose. The estimate on the total curvature is also given in the lemma; notice that sliding a loop between t_0 and t_1 does not affect total curvature.

Assume now that $J > 1$. Let $W \subset U_J$ be an open set whose closure is contained in U_J and such that $K = W \cup \bigcup_{1 \leq j \leq J-1} U_j$. We now slide the loop in t_J to position t_{J-1} in W, allowing for the loop to stop elsewhere for $p \in U_J \smallsetminus W$. More precisely, let $u : K \to [0,1]$ be a continuous function with $u(p) = 1$ for $p \in W$ and $u(p) = 0$ for $p \notin U_J$. Define $H_J : [0,1] \times K \to \mathcal{L}_Q$ by

$$H_J(s,p) = \begin{cases} f(p), & p \notin U_J, \\ g_J(p)^{[((1-u(p)s)t_J(p)+u(p)st_{J-1}(p))\#2]}, & p \in U_J. \end{cases}$$

Notice that $\operatorname{tot}(H_J(s,p)) = \operatorname{tot}(f(p))$. Let $\hat{f}(p) = H_J(1,p)$, $\hat{U}_j = U_j$ for $j < J-1$ and $\hat{U}_{J-1} = U_{J-1} \cup W$; the hypotheses of the Lemma apply to $\hat{f}$ with a smaller value of J and therefore $\hat{f}$ is loose. By Lemma 6.5, so is f. The estimate on total curvature also follows. ■

7 Multiconvex curves

A curve $\gamma \in \mathcal{L}_z$ is *multiconvex* of multiplicity k if there exist $0 = t_0 < t_1 < \cdots < t_k = 1$ such that

(a) $\mathfrak{F}_\gamma(t_i) = I$ for $i < k$;

(b) the restrictions $\gamma|_{[t_{i-1},t_i]}$ are convex arcs for $1 \leq i \leq k$.

Notice that for $i < k$ these restrictions are then simple closed curves (see Figure 3). Let $\mathcal{M}_k \subset \mathcal{L}_z$ be the set of multiconvex curves of multiplicity k.

Notice that ν_k is multiconvex of multiplicity k. A curve $\gamma \in \mathcal{L}_z$ is multiconvex of multiplicity 1 if and only if it is convex, so that for $z = -\mathbf{1}$ we have $\mathcal{M}_1 = \mathcal{L}_{-\mathbf{1},c}$. By Lemma 4.1, if $\gamma \in \mathcal{M}_k$ then $2(k-1)\pi < \operatorname{tot}(\gamma) < 4k\pi$. It is easy to see that, for k odd, $\mathcal{M}_k \neq \emptyset$ if and only if z is convex; similarly, for k even, $\mathcal{M}_k \neq \emptyset$ if and only if $-z$ is convex.

In [18], other submanifolds $\mathcal{F}_k \subset \mathcal{L}$ (of *flowers* of order k, or of $2k-1$ petals) are introduced which play a role somewhat similar to $\mathcal{M}_k$. For results up to this point, it is indeed largely a matter of taste to use multiconvex curves or flowers. For the final part of the paper, however, multiconvex curves work better.

Lemma 7.1 *Let $z \in \mathbb{S}^3$. Let k be a positive integer. The closed subset $\mathcal{M}_k \subset \mathcal{L}_z$ (if non-empty) is a contractible submanifold of codimension $2k-2$ with trivial normal bundle.*

Proof: Assume that $-z$ is convex. Consider the geodesic $\rho \subset \mathbb{S}^2$ passing through $\pm e_1$ and $\pm e_3$ (where $e_1 = (1,0,0)$). After a projective transformation we may assume that any convex curve $\gamma \in \mathcal{L}_{-z}$ crosses ρ transversally once for some $t \in (0,1)$.

We first define open sets $U_k \subset \mathcal{L}_{(\mathbf{-1})^k z}$. A curve $\gamma \in \mathcal{L}_{(\mathbf{-1})^k z}$ belongs to U_k if and only if:

(a) all intersections between γ and ρ are transversal;

(b) there are exactly $2k$ values

$$0 = t_0 < t_{\frac{1}{2}} < t_1 < \cdots < t_{k-1} < t_{k-\frac{1}{2}} < 1$$

of $t \in [0,1)$ for which $\gamma(t) \in \rho$;

(c) consecutive intersections $\gamma(t_j)$ and $\gamma(t_{j\pm\frac{1}{2}})$ are distinct;

(d) arcs of γ between t_j and $t_{j+\frac{1}{2}}$, $t_{j+\frac{1}{2}}$ and t_{j+1} or $t_{k-\frac{1}{2}}$ and 1 are convex.

Notice that U_k is indeed open and $t_j : U_k \to [0,1]$ are continuous functions. We may continuously define the arguments θ_j of the points $\gamma(t_j(\gamma))$ by

$$\gamma(t_j(\gamma)) = \cos(\theta_j(\gamma))e_1 + \sin(\theta_j(\gamma))e_3, \quad \theta_0(\gamma) = 0,$$

and, for integer j,

$$\theta_j < \theta_{j+\frac{1}{2}} < \theta_j + \pi, \quad \theta_{j+\frac{1}{2}} > \theta_{j+1} > \theta_{j+\frac{1}{2}} - \pi.$$

Also, $\langle \gamma'(t_j(\gamma)), e_2 \rangle$ is positive if j is an integer (and negative otherwise). For j an integer, we continuously define the argument η_j of $\gamma'(t_j(\gamma))$ by

$$\gamma'(t_j(\gamma)) = r\left(\cos(\eta_j(\gamma))e_2 + \sin(\eta_j(\gamma))n\right),$$
$$n = -\sin(\theta_j(\gamma))e_1 + \cos(\theta_j(\gamma))e_3, \quad -\pi/2 < \eta_j(\gamma) < \pi/2, \quad r > 0.$$

Now define $M_k : U_k \to \mathbb{R}^{2k-2}$ by

$$M_k(\gamma) = (\theta_1(\gamma), \eta_1(\gamma), \theta_2(\gamma), \eta_2(\gamma), \ldots, \theta_{k-1}(\gamma), \eta_{k-1}(\gamma)).$$

The smooth map M_k is a submersion and $\mathcal{M}_k$ is the inverse image of $0 \in \mathbb{R}^{2k-2}$. This proves that $\mathcal{M}_k$ is a smooth submanifold of codimension $2k-2$ and trivializes its normal bundle.

Finally, we must prove that $\mathcal{M}_k$ is contractible. The fact that $\mathcal{L}_{-z,c} = \mathcal{M}_1(z)$ is contractible is well known ([1] and [25], Lemma 5). The subset $\hat{\mathcal{M}}_k$ of $\mathcal{M}_k$ of curves for which $t_j = j/k$ is naturally identified with $(\mathcal{M}_1(-\mathbf{1}))^{(k-1)} \times (\mathcal{M}_1(z))$ (reparametrize $\gamma|_{[t_j,t_{j+1}]}$ to define $\gamma_j \in \mathcal{M}_1$) and therefore is also contractible. But $\hat{\mathcal{M}}_k \subset \mathcal{M}_k$ is a deformation retract: just use piecewise affine functions to reparametrize each curve so that $t_j = j/k$ (for all j). ■

The previous result allows us to use each $\mathcal{M}_k$ to define an element $m_{2k-2} \in H^{2k-2}(\mathcal{L}_z;\mathbb{Z})$ by counting intersections with $\mathcal{M}_k$. For de Rham cohomology, for instance, we consider Thom's form in the (trivial) normal bundle to $\mathcal{M}_k$ and use the identification of this bundle with a tubular neighborhood of $\mathcal{M}_k$ to define a closed $(2k-2)$-form ω which is a representative of m_{2k-2}. Thus, if $f : K \to \mathcal{L}_z$ is a smooth map from an oriented compact $(2k-2)$-dimensional manifold K to $\mathcal{L}_z$ which is transversal to $\mathcal{M}_k$ then the integral of the pull back of ω over K equals the number of intersections of f with $\mathcal{M}_k$, counted with sign. If the map is not smooth or not (topologically) transversal we may perturb it so that it becomes both smooth and transversal: the number of intersections is still well defined. We denote this integer by $m_{2k-2}(f)$. The elements $m_{2k-2} \in H^{2k-2}(\mathcal{L}_z)$ will turn out to be the "extra" cohomology (as compared to $\mathcal{I}_z$); compare with Corollary 1.2.

As we shall see in Lemma 7.2 below, the map g_0 (introduced in Section 5) is tight and satisfies $m_2(g_0) = \pm 1$ but $m_2(g_0^{[t_0\#2]}) = 0$. For $N \in H^2(\mathcal{L}_{+\mathbf{1}};\mathbb{Z})$ as defined in Proposition 5.3, we have $N(g_0) = N(g_0^{[t_0\#2]}) = \pm 1$ (from Proposition 5.3 and Lemma 6.1). It follows that m_2 and N span a copy of $\mathbb{Z}^2$ in $H^2(\mathcal{L}_{+\mathbf{1}};\mathbb{Z})$ and that g_0 and $g_0^{[t_0\#2]}$ span a copy of $\mathbb{Z}^2$ in $\pi_2(\mathcal{L}_{+\mathbf{1}}) = H_2(\mathcal{L}_{+\mathbf{1}};\mathbb{Z})$. Compare this with Corollary 1.2: we shall later see that m_2 and N actually span $H^2(\mathcal{L}_{+\mathbf{1}};\mathbb{Z})$ and that g_0 and $g_0^{[t_0\#2]}$ actually span $\pi_2(\mathcal{L}_{+\mathbf{1}})$.

In the following lemma the sphere $\mathbb{S}^2$ will be the compact manifold (usually called K) in the domain of a map. Let $\mathbf{s} = -e_3$, $\mathbf{n} = e_3$ be the south and north pole, respectively. The base point of $\mathbb{S}^2$ is $\mathbf{s}$.

Lemma 7.2 *There exist maps $g_s : \mathbb{S}^2 \to \mathcal{L}_{\mathbf{1}}$, $s \in [0, \frac{1}{2})$, such that:*

(a) $g_s(\mathbf{s}) = \nu_2$ and $g_s(\mathbf{n})$ is a reparametrization of ν_4;

(b) if $g_s(p)$ is multiconvex then $p = \mathbf{s}$ or $p = \mathbf{n}$;

(c) g_s is topologically transversal to $\mathcal{M}_2$ at $\mathbf{s}$;

(d) if $t \in [1-s, 1]$ then $(g_s(p))(t) = \nu_2(t)$;

(e) if $\tilde{\mathfrak{F}}_{g_s(p)}(t) \in \mathcal{C}_{\hat{\mathbf{k}}}$, $t \in (0,1)$, then either $p = \mathbf{s}$, $p = \mathbf{n}$ or $t \in [1-s, 1]$;

(f) given $t \in (0, 1-s)$, the map $p \mapsto \tilde{\mathfrak{F}}_{g_s(p)}(t)$ is topologically transversal to $\mathcal{C}_{\hat{\mathbf{k}}}$;

(g) the maps g_s are all homotopic to g_0;

(h) the maps g_s satisfy $m_2(g_s) = \pm 1$ and are all tight.

We reiterate that the map g_0 is the same one introduced in Section 5. The maps g_s for $s > 0$ may be informally described as modifications of g_0 by forcing g_s to coincide with ν_2 for $t \in [1-s, 1]$ (as in item (d)). The proof of the lemma thus splits into three parts: we first prove that g_0 satisfies all the desired properties, we next construct g_s for $s > 0$ and we finally verify that the properties remain true for $s > 0$. More precisely, we first construct g_s for small positive s by making small adjustments to g_0; for large s we use projective transformations. The constructions of g_s are rather explicit, and it should be noted that many arbitrary choices are made during the construction, sometimes in order to facilitate some later argument. The maps g_s will be the building blocks in the construction of the maps h_{2k-2} in Lemma 7.3 below.

The sign ambiguity in the last item comes from the fact that we were not too careful to define either a standard transversal orientation to $\mathcal{M}_2$ or a standard orientation for $K = \mathbb{S}^2$, the domain of g_s. Recall that $\mathcal{C}_{\hat{\mathbf{k}}} \subset \mathbb{S}^3$ is a subgroup defined before Lemma 5.2.

Proof: We first prove that g_0 satisfies the desired properties. Items (a) and (d) are immediate and item (e) follows directly from Lemma 5.2. Item (b) is a direct consequence of item (e). For items (c) and (f) we first notice that we are talking about isolated points. Indeed, from (b), $p = \mathbf{s}$ is the only point for which $g_0(p) \in \mathcal{M}_2$. Similarly, from (e), $p = \mathbf{s}$ is the only point for which $\mathfrak{F}_{g_0(p)}(t) \in \mathcal{C}_{\hat{\mathbf{k}}}$. In either case we need to study what happens when p goes around $\mathbf{s}$, drawing a small circle. For (c), we need to prove that $g_0(p)$ will go around $\mathcal{M}_2$ once, or, equivalently, that $M_2(g_0(p))$ will go around the origin once. For (f), we need to prove that $\mathfrak{F}_{g_0(p)}(t)$ will go around the circle $\mathcal{C}_{\hat{\mathbf{k}}}$ once. Notice that both observations are rather clear from Figure 6.

Go back to the topological transversality argument in Proposition 5.3. Recall that $(B_0(t; t + \frac{1}{2}))^{-1} B_0^{\bullet}(t; t + \frac{1}{2})$ performs one full turn around the origin when t goes from 0 to 1/3.

Let $p = (\cos\theta \sin\alpha, \sin\theta \sin\alpha, -\cos\alpha)$; let M_2 and U_2 be as in Lemma 7.1. For sufficiently small α, we have $g_0(p) \in U_2$. From the above computations, for sufficiently small α, $M_2(g_0(p))$ also performs a full turn around the origin when θ goes from 0 to 2π, completing the proof of (c). Item (f) follows similarly from these computations for $t = 1/2$; we notice that, by continuity, the number of turns of $\tilde{\mathfrak{F}}_{g_0(p)}(t)$ around $\mathcal{C}_{\hat{\mathbf{k}}}$ must be constant; this completes the proof of item (f).

Notice now that items (a) through (f) imply that g_0 intersects $\mathcal{M}_2$ topologically transversally and exactly once. Thus, for $m_2 \in H^2$ as above, $m_2(g_0) =$

± 1. On the other hand, $g_0^{[t_0\#2]}$ can not possibly intersect $\mathcal{M}_2$ and therefore $m_2(g_0^{[t_0\#2]}) = 0 \neq m_2(g_0)$. Thus g_0 and $g_0^{[t_0\#2]}$ are not homotopic and g_0 is therefore tight. This completes the proof of item (h), of the case $s = 0$ and of the remarks preceding the statement of the lemma.

We now construct g_s for $s > 0$, s small. By compactness, there exists $\epsilon_1 > 0$ such that for all $p \in \mathbb{S}^2$, the arc $g_0(p)|_{[-2\epsilon_1,+2\epsilon_1]}$ is convex. Here we interpret $g_0(p)$ as a 1-periodic function from $\mathbb{R}$ to $\mathbb{S}^2$. Recall that $\tilde{\mathfrak{F}}_{g_0(p)}(0) = \tilde{\mathfrak{F}}_{\nu_2}(0) = \mathbf{1}$. Again by compactness, there exists $\epsilon_2 \in (0, \epsilon_1/4)$ such that, for all $t \in [0, \epsilon_2]$ and for all $p \in \mathbb{S}^2$, we have

$$(\mathfrak{F}_{\nu_2}(t))^{-1}\, \mathfrak{F}_{g_0(p)}(\epsilon_1), \big(\mathfrak{F}_{g_0(p)}(-\epsilon_1)\big)^{-1}\, \mathfrak{F}_{\nu_2}(-t) \in \mathrm{Bru}_{(13);2}\,.$$

We want to define $\hat{g} : \mathbb{S}^2 \to \mathcal{L}_{\mathbf{1}}$ with

$$(\hat{g}(p))(t) = \begin{cases} \nu_2(t), & t \in [0, \epsilon_2] \cup [1 - \epsilon_2, 1], \\ (g_0(p))(t), & t \in [\epsilon_1, 1 - \epsilon_1]. \end{cases}$$

As in Lemma 3.1, for each $p \in \mathbb{S}^2$, there exist ellipses $\mathcal{E}_+$ and $\mathcal{E}_-$ and parametrizations by projective arc-length $\gamma_+ : [\epsilon_2, \epsilon_1] \to \mathcal{E}_+ \subset \mathbb{S}^2$ and $\gamma_- : [1 - \epsilon_1, 1 - \epsilon_2] \to \mathcal{E}_- \subset \mathbb{S}^2$ such that

$$\begin{gathered} \gamma_+(\epsilon_2) = \nu_2(\epsilon_2), \quad \gamma_+'(\epsilon_2) = \nu_2'(\epsilon_2), \quad \mathfrak{F}_{\gamma_+}(\epsilon_1) = \mathfrak{F}_{g(p)}(\epsilon_1), \\ \gamma_-(1-\epsilon_2) = \nu_2(1-\epsilon_2), \quad \gamma_-'(1-\epsilon_2) = \nu_2'(1-\epsilon_2), \quad \mathfrak{F}_{\gamma_-}(1-\epsilon_1) = \mathfrak{F}_{g(p)}(1-\epsilon_1), \end{gathered}$$

and, furthermore, $\mathcal{E}_+$ and $\mathcal{E}_-$ osculate the circle $\mathcal{C}_{\hat{\mathbf{k}}}$ at $\nu_2(\epsilon_2)$ and $\nu_2(1 - \epsilon_2)$, respectively. The ellipses and parametrizations are uniquely and continuously defined. Complete the definition of $\hat{g}$ by

$$(\hat{g}(p))(t) = \begin{cases} \gamma_+(t), & t \in [\epsilon_2, \epsilon_1], \\ \gamma_-(t), & t \in [1 - \epsilon_1, 1 - \epsilon_2]. \end{cases}$$

Notice that since 5 points define a conic, the ellipses $\mathcal{E}_\pm$ have no tangency point to $\mathcal{C}_{\hat{\mathbf{k}}}$ besides $\nu_2(\pm\epsilon_2)$.

For $s \leq 2\epsilon_2$, set $g_s(p)(t) = (\mathfrak{F}_{\nu_2}(\epsilon_2))^{-1}\hat{g}(t - \epsilon_2)$. We claim that the function g_s has all the required properties. Item (a) is obvious. Item (d) holds by construction and item (e) follows from the last observation in the previous paragraph; item (b) now follows. Topological transversality (items (c) and (f)) is handled as for $s = 0$. Item (g) follows either from an explicit computation or from the contractibility of $\mathcal{L}_{z,c}$ and item (h) now follows. This completes the proof of the theorem for $s \leq 2\epsilon_2$.

For $c \in \mathbb{R}$, let

$$A(c) = \begin{pmatrix} 1 & c & c^2/2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \exp\left(c \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Notice that $\pi(A(c))e_1 = e_1$, $\pi(A(c))I = I$ and $\pi(A(c))$ takes the circle $\mathcal{C}_{\hat{\mathbf{k}}}$ to itself. Let $\hat{g}_c : \mathbb{S}^2 \to \boldsymbol{\mathcal{L}}_{\mathbf{1}}$ be given by $\hat{g}_c(p) = \pi(A(c)) \circ g_{\epsilon_2}(p)$. Given c, each arc $(\hat{g}_c(p))_{[1-\epsilon_2,1]}$ is a fixed parametrization of an arc of $\mathcal{C}_{\hat{\mathbf{k}}}$. Changing c, that arc can be taken to have any required length. In other words, given $s \in (\epsilon_2, \frac{1}{2})$ there exists a unique $c \in \mathbb{R}$ for which $\mathfrak{F}_{\hat{g}_c(p)}(1-\epsilon_2) = \mathfrak{F}_{\nu_2}(1-s)$: define g_s by suitably reparametrizing this $\hat{g}_c$. All the required properties follow by construction. ■

Let $z \in \mathbb{S}^3$ with $-z$ convex. We are now ready to construct tight maps $h_{2k-2} : \mathbb{S}^{2k-2} \to \mathcal{L}_{(-\mathbf{1})^k z}$ corresponding to the spheres attached to $\mathcal{I}_{\Pi(z)}$ to obtain $\mathcal{L}_{\Pi(z)}$ as in Theorems 1 and 2. Alternatively, h_{2k-2} define the "extra" generators of $H_*(\mathcal{L}_{\Pi(z)};\mathbb{Z})$ (compared to $H_*(\mathcal{I}_{\Pi(z)};\mathbb{Z})$); see Corollary 1.2.

Lemma 7.3 *Let $k > 1$ be a positive integer. Let $z \in \mathbb{S}^3$ with $-z$ convex. There exist tight maps $h_{2k-2} : \mathbb{S}^{2k-2} \to \mathcal{L}_{(-\mathbf{1})^k z}$ which:*

(a) intersect $\mathcal{M}_k$ exactly once and topologically transversally;

(b) do not intersect $\mathcal{M}_{k'}$ for $k' \neq k$;

(c) satisfy $m_{2k-2}(h_{2k-2}) = \pm 1$;

(d) are homotopic to a constant as maps $\mathbb{S}^{2k-2} \to \mathcal{I}_{(-\mathbf{1})^k z}$.

Proof: Let $\mathbb{D}^2 \subset \mathbb{R}^2$ be the closed disk of radius 1. We first construct a function $\hat{h} : (\mathbb{D}^2)^{(k-1)} \to \mathcal{L}_{(-\mathbf{1})^k z}$. Consider $\epsilon_0 > 0$ such that if $t \in (0, \epsilon_0)$ then $-(\tilde{\mathfrak{F}}_{\nu_1}(kt))^{-1} z$ is convex. Let $s_0 = k(1+\epsilon_0)$ and $z_0 = \tilde{\mathfrak{F}}_{\nu_{s_0}}(\frac{1}{k}) = -\tilde{\mathfrak{F}}_{\nu_1}(\epsilon_0)$. We shall have $(\hat{h}(0))(t) = \nu_{s_0}(t)$ for $t \in [0, \frac{k-1}{k}]$ and $\tilde{\mathfrak{F}}_{\hat{h}(p)}(\frac{i}{k}) = z_0^i$ for all p and for all integers $i < k$ (in other words, we are starting the definition here; the expression "we shall have" is to be understood as: "here is yet another property of $\hat{h}$, which is clearly consistent with what we demanded before").

Let $\gamma_k : [\frac{k-1}{k}, 1] \to \mathbb{S}^2$ be a convex arc with $\tilde{\mathfrak{F}}_{\gamma_k}(\frac{k-1}{k}) = z_0^{k-1}$, $\tilde{\mathfrak{F}}_{\gamma_k}(1) = (-\mathbf{1})^k z$. We shall have $(\hat{h}(p))(t) = \gamma_k(t)$ for all $p \in (\mathbb{D}^2)^{(k-1)}$ and $t \in [\frac{k-1}{k}, 1]$. Let $s_1 = 1 - \frac{s_0}{2k} = \frac{1-\epsilon_0}{2}$ and let g_{s_1} be as in Lemma 7.2 so that $\tilde{\mathfrak{F}}_{g_{s_1}(p)}(1-s_1) = z_0$ for all $p \in \mathbb{S}^2$. Recall that $\tilde{\nu}_4 = g_{s_1}(\mathbf{n})$ is a reparametrization of ν_4 with $\tilde{\nu}_4(t) = \nu_2(t)$ for all $t > 1 - s_1$. Define $w : \mathbb{D}^2 \to \mathbb{S}^2$ by

$$w(r\cos\theta, r\sin\theta) = \begin{cases} (\cos\theta \sin(4r), \sin\theta\sin(4r), -\cos(4r)), & r \leq \pi/4, \\ (0,0,1), & r \geq \pi/4. \end{cases}$$

Consider $p = (p_1, p_2, \ldots, p_{k-1}) \in (\mathbb{D}^2)^{(k-1)}$, $p_i \in \mathbb{D}^2$. For $t \in [\frac{i-1}{k}, \frac{i}{k}]$ let $t_i = (1-s_1)k\left(t - \frac{i-1}{k}\right)$; if $|p_i| \leq \frac{\pi}{4}$ we shall have $(\hat{h}(p))(t) = \Pi(z_0^{i-1})g_{s_1}(w(p_i))(t_i)$. Let $\tilde{\nu}_{8k}$ be a reparametrization of ν_{8k} with $\tilde{\nu}_{8k}(t) = \nu_2(t)$ for all $t > 1 - s_1$. Let $\tilde{g} : [\frac{\pi}{4}, \frac{7}{8}] \to \boldsymbol{\mathcal{L}}_{+\mathbf{1}}$ be a path from $\tilde{g}(\frac{\pi}{4}) = \tilde{\nu}_4$ to $\tilde{g}(\frac{7}{8}) = \tilde{\nu}_{8k}$ satisfying $\tilde{g}(\tau)(t) = \nu_2(t)$

for all $t > 1 - s_1$; notice that $\tilde{\mathfrak{F}}_{\tilde{g}(\tau)}(1-s_1) = z_0$ for all $\tau \in [\frac{\pi}{4}, \frac{7}{8}]$. For $t \in [\frac{i-1}{k}, \frac{i}{k}]$ and t_i as above we shall have

$$(\hat{h}(p))(t) = \begin{cases} \Pi(z_0^{i-1})\tilde{g}(|p_i|)(t_i), & |p_i| \in [\frac{\pi}{4}, \frac{7}{8}], \\ \Pi(z_0^{i-1})\tilde{\nu}_{8k}(t_i), & |p_i| \geq \frac{7}{8}, \end{cases}$$

completing the construction of $\hat{h}$.

From Lemma 7.2, $\hat{h}(0) \in \mathcal{M}_k$, $\hat{h}(p) \in \mathcal{M}_k$ implies $p = 0$ and $\hat{h}$ is topologically transversal to $\mathcal{M}_k$, with a single intersection at $p = 0$. Consider

$$\hat{h}_1 = \hat{h}|_{\partial((\mathbb{D}^2)^{(k-1)})} : \partial((\mathbb{D}^2)^{(k-1)}) \to \mathcal{L}_{(-\mathbf{1})^k z}.$$

We apply Lemma 6.6 to prove that $\hat{h}_1$ is loose: here $K = \partial((\mathbb{D}^2)^{(k-1)})$ (which is homeomorphic to $\mathbb{S}^{(2k-3)}$), $J = k-1$, $t_j = \frac{j}{k} - \frac{1}{2k}$ and $(p_1, p_2, \ldots, p_{k-1}) \in U_j$ if $i \neq j$ implies $|p_i| > \frac{7}{8}$. There exists therefore a homotopy $H : [0,1] \times \partial((\mathbb{D}^2)^{(k-1)}) \to \mathcal{L}_{(-\mathbf{1})^k z}$ with $H(0,p) = \hat{h}_1(p)$ and $H(1,p) = (\hat{h}_1(p))^{[t_k \# 2]}$, $t_k = 1 - \frac{1}{2k}$.

The homotopy H may be assumed to be disjoint from $\mathcal{M}_k$. In order to see this we give estimates on the total curvature. The total curvature of $\tilde{\nu}_{8k}$ equals $16k\pi$, and the total curvature of its restriction to $[0, 1-s_1]$ is greater than $(16k-2)\pi$. If $p \in \partial((\mathbb{D}^2)^{(k-1)})$ we have at least one index j for which $|p_j| = 1$; for such j, the total curvature in the interval $[\frac{j-1}{k}, \frac{j}{k}]$ is greater than $(16k-2)\pi$. The total curvature of $\hat{h}(p)$ is therefore greater than $(16k-2)\pi$. By Lemma 6.6, we may construct H as above with $\operatorname{tot}(H(s,p)) > (16k-2)\pi > 4k\pi$ (for all $s \in [0,1]$ and all $p \in \partial((\mathbb{D}^2)^{(k-1)})$) and therefore H is disjoint from $\mathcal{M}_k$, as claimed.

Let $\mathbb{D}^{(2k-2)}$ be the closed disk of dimension $2k-2$ and radius 1. There exists a homeomorphism from $\mathbb{D}^{(2k-2)}$ to $(\{0\} \times (\mathbb{D}^2)^{(k-1)}) \cup ([0,1] \times \partial((\mathbb{D}^2)^{(k-1)}))$. Compose this homeomorphism with $\hat{h}$ and H to define a map $\tilde{h} : \mathbb{D}^{(2k-2)} \to \mathcal{L}_{(-\mathbf{1})^k z}$ with $\tilde{h}(0) \in \mathcal{M}_k$; $\tilde{h}(p) \in \mathcal{M}_k$ implies $p = 0$; $\tilde{h}$ is topologically transversal to $\mathcal{M}_k$, with a single intersection at $p = 0$; $\operatorname{tot}(\tilde{h}(p)) > 16k\pi$ for all $p \in \mathbb{S}^{(2k-3)}$. We furthermore have $\tilde{h}(p) = \gamma^{[t_k \# 2]}$ for all $p \in \mathbb{S}^{(2k-3)}$ (for some γ). More precisely, let $Q_k = \mathfrak{F}_{\gamma_k}(t_k)$; after a reparametrization we may assume that, for all $p \in \mathbb{S}^{(2k-3)}$, $(\tilde{h}(p))(t_k + \epsilon_1 \tau) = Q_k \nu_2(\tau + \frac{1}{2})$ for $|\tau| \leq \frac{1}{2}$, where $\epsilon_1 \in (0, \frac{1}{4k})$ is a small positive constant.

The sphere $\mathbb{S}^{(2k-2)}$ is homeomorphic to $S = (\{0,1\} \times \mathbb{D}^{(2k-2)}) \cup ([0,1] \times \mathbb{S}^{(2k-3)})$. Let $\gamma_2 : [0,1] \to \mathcal{L}_{\mathbf{1}}$ with $\gamma_2(0) = \nu_2$, $\gamma_2(1) = \nu_4$. Define $\tilde{h}_{2k-2} : S \to \mathcal{L}_{(-\mathbf{1})^k z}$ by

$$\tilde{h}_{2k-2}(s,p)(t) = \begin{cases} (\tilde{h}(p))(t), & t \notin (t_k - \frac{\epsilon_1}{2}, t_k + \frac{\epsilon_1}{2}), \\ (\tilde{h}(p))(t), & s = 0, \\ (\tilde{h}(p))^{[t_k \# 2]}(t), & s = 1, \\ Q_k \gamma_2(s)(\tau + \frac{1}{2}), & |p| = 1,\ t = t_k + \epsilon_1 \tau,\ \tau \in [-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

We claim that $\tilde{h}_{2k-2}$ is homotopic to a point in $\mathcal{I}_{(-\mathbf{1})^k z}$, or, equivalently (Proposition 6.4), that $\tilde{h}_{2k-2}^{[t_k\#2]}$ can be extended to a map from $[0,1]\times\mathbb{D}^{(2k-2)}$ to $\mathcal{L}_{(-\mathbf{1})^k z}$. Indeed, after a reparametrization, $\tilde{h}^{[t_k\#2]}:\mathbb{D}^{(2k-2)}\to\mathcal{L}_{(-\mathbf{1})^k z}$ may be assumed to satisfy $(\tilde{h}^{[t_k\#2]}(p))(t_k+\epsilon_1\tau)=\tilde{Q}(p)\nu_2(\tau+\frac{1}{2})$ for all $p\in\mathbb{D}^{(2k-2)}$, where $\tau\in[-\frac{1}{2},\frac{1}{2}]$ and $\tilde{Q}:\mathbb{D}^{(2k-2)}\to SO_3$ satisfies $\tilde{Q}(p)=Q_k$ if $|p|=1$. Define $\bar{h}:[0,1]\times\mathbb{D}^{(2k-2)}\to\mathcal{L}_{(-\mathbf{1})^k z}$ by

$$\bar{h}(s,p)(t)=\begin{cases}(\tilde{h}^{[t_k\#2]}(p))(t), & t\notin(t_k-\frac{\epsilon_1}{2},t_k+\frac{\epsilon_1}{2}),\\ \tilde{Q}(p)\gamma_2(s)(\tau+\frac{1}{2}), & t=t_k+\epsilon_1\tau,\ \tau\in[-\frac{1}{2},\frac{1}{2}];\end{cases}$$

up to reparametrization, $\bar{h}$ is the desired extention.

Identifying S with $\mathbb{S}^{2k-2}$, the function $\tilde{h}_{2k-2}:\mathbb{S}^{2k-2}\to\mathcal{L}_{(-\mathbf{1})^k z}$ thus satisfies item (d). By construction, the only multiconvex curves in its image are $\nu_k=\tilde{h}_{2k-2}(0,0)$ and $\tilde{h}_{2k-2}(1,0)$, which is a reparametrization of ν_{k+2}. Define h_{2k-2} by perturbing $\tilde{h}_{2k-2}$ near $(1,0)$ so as to avoid $\mathcal{M}_{k+2}$; by transversality, this can be done: the codimension of $\mathcal{M}_{k+2}$ is larger than the dimension of $\mathbb{S}^{2k-2}$. Item (b) is therefore satisfied. Topological transversality in item (a) also follows by construction and by items (c) and (f) of Lemma 7.2. Finally, $m_{2k-2}(h_{2k-2})=\pm 1$ follows from items (a) and (b), proving item (c). ■

The following corollary sums up some of the topological differences between the spaces $\mathcal{L}_z$ and $\mathcal{I}_z$ which we have proved in this section.

Corollary 7.4 *Consider $z\in\mathbb{S}^3$ with $-z$ convex. For $k\geq 1$, the elements $m_{2k-2}\in H^{2k-2}(\mathcal{L}_{(-\mathbf{1})^k z})$ do not belong to the image of $i^*=H^*(i):H^*(\mathcal{I}_{(-\mathbf{1})^k z})\to H^*(\mathcal{L}_{(-\mathbf{1})^k z})$. The maps $h_{2k-2}:\mathbb{S}^{2k-2}\to\mathcal{L}_{(-\mathbf{1})^k z}$ define non-zero elements in the kernel of $i_*=\pi_{2k-2}(i):\pi_{2k-2}(\mathcal{L}_{(-\mathbf{1})^k z})\to\pi_{2k-2}(\mathcal{I}_{(-\mathbf{1})^k z})$.*

The aim of the rest of the paper is to prove that these are, in a sense, the *only* differences.

Our final lemma in this section is an easy consequence of the previous results and will be used later.

Lemma 7.5 *Let B be a compact manifold of dimension $n+1$ with boundary $\partial B=K$. Let $f_0:B\to\mathcal{L}_z$ be a continuous map with $f_0|_K$ disjoint from all $\mathcal{M}_k$. Then there exists a continuous map $f_1:B\to\mathcal{L}_z$ with $f_0|_K=f_1|_K$ and f_1 disjoint from all $\mathcal{M}_k$.*

Proof: For each k, let $V_k\subset\mathcal{L}_z$ be a closed tubular neighborhood of $\mathcal{M}_k$. We may assume the sets V_k to be disjoint and $f_0|_K$ to be disjoint from the sets V_k. We may furthermore assume the map $h_{2k-2}:\mathbb{S}^{2k-2}\to\mathcal{L}_z$ to be topologically transversal to ∂V_k so that $D_k=h_{2k-2}^{-1}(V_k)\subset\mathbb{S}^{2k-2}$ is a disk around the south pole. Since $\mathcal{M}_k$ is a contractible Hilbert manifold, $\mathcal{M}_k$ is homeomorphic to the

Hilbert space $\mathcal{H}$. Let $\mathbb{D}^{2k-2} \subset \mathbb{R}^{2k-2}$ be the closed unit ball of radius 1. Let $\psi_k = (\psi_{k,1}, \psi_{k,2}) : V_k \to \mathcal{H} \times \mathbb{D}^{2k-2}$ be a homeomorphism taking $\mathcal{M}_k$ to $\mathcal{H} \times \{0\}$; we may assume that $\psi_{k,1} \circ h_{2k-2}$ is constant equal to 0 and that $\phi_k = \psi_{k,2} \circ h_{2k-2}$ is a homeomorphism between D_k and $\mathbb{D}^{2k-2}$. For $c \in \mathbb{R}$ and $\gamma \in V_k$, let $\mu(c,\gamma) \in V_k$ be such that $\psi_{k,1}(\mu(c,\gamma)) = c\psi_{k,1}(\gamma)$ and $\psi_{k,2}(\mu(c,\gamma)) = \psi_{k,2}(\gamma)$.

Let $\tilde{D}_k \subset D_k$ be the inverse image under ϕ_k of the closed disk of radius $1/2$. Let $\rho : \tilde{D}_k \to \mathbb{S}^{2k-2}$ be a continuous map coinciding with the identity on $\partial\tilde{D}_k$ and avoiding the south pole. Define

$$f_1(p) = \begin{cases} f_0(p), & f_0(p) \notin \bigcup_k V_k, \\ \mu(2|\psi_{k,2}(f_0(p))| - 1, f_0(p)), & f_0(p) \in V_k, |\psi_{k,2}(f_0(p))| \geq \frac{1}{2}, \\ h_{2k-2}(\rho(\phi_k^{-1}(\psi_{k,2}(f_0(p))))), & f_0(p) \in V_k, |\psi_{k,2}(f_0(p))| \leq \frac{1}{2}. \end{cases}$$

The map f_1 satisfies the required conditions. ■

8 Grafting

In this section we introduce the process of *grafting* curves; similar ideas are considered in [22] and [27].

Subintervals $[t_0, t_1] \subset (0,1)$ will be called *arcs*: an arc $[t_0, t_1]$ is *graftable* for a locally convex curve γ if there exists a projective transformation taking γ to γ_1 and real numbers $\theta_0, \theta_1, \phi_0, \phi_1$ with $-\pi/2 < \phi_0 < 0 < \phi_1 < \pi/2$ and

$$\tilde{\mathfrak{F}}_{\gamma_1}(t_0) = e^{\theta_0 \mathbf{k}/2} e^{\phi_0 \mathbf{j}/2}, \quad \tilde{\mathfrak{F}}_{\gamma_1}(t_1) = e^{\theta_1 \mathbf{k}/2} e^{\phi_1 \mathbf{j}/2} e^{\pi \mathbf{i}/2}. \tag{2}$$

Recall that $\Pi(e^{\theta \mathbf{k}/2})$ is a rotation by θ around the z axis:

$$\Pi(e^{\theta \mathbf{k}/2}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

similarly, $\Pi(e^{\phi \mathbf{j}/2})$ is a rotation by ϕ around the y axis.

Translating into a more geometric language, equation 2 says that γ_1 is tangent at t_i to the circle $z = -\sin(\phi_i)$; in both cases the orientation of γ_1 is consistent with a locally convex orientation of the circles.

Notice that the existence of the desired projective transformation depends only on the value of $\mathfrak{F}_\gamma(t_0; t_1)$. A matrix $Q \in SO_3$ is *graftable* if $\mathfrak{F}_\gamma(t_0; t_1) = Q$ implies that the arc $[t_0, t_1]$ is graftable for γ.

Lemma 8.1 *Let $Q \in SO_3$: Q is graftable if and only if Q belongs to one of the Bruhat cells below:*

$$\mathrm{Bru}_{(13);1}, \mathrm{Bru}_{(13);4}, \mathrm{Bru}_{(13);7}, \mathrm{Bru}_{(123);3}, \mathrm{Bru}_{(123);5}, \mathrm{Bru}_{(132);5}, \mathrm{Bru}_{(132);6}, \mathrm{Bru}_{e;5}\,.$$

Proof: The definition is clearly invariant under projective transformation and therefore if Q_1 and Q_2 belong to the same Bruhat cell then Q_1 is graftable if and only if Q_2 is.

We need therefore to check which Bruhat cells are touched by

$$\mathfrak{F}_{\gamma_1}(t_0;t_1) = e^{-\phi_0 \mathbf{j}/2} e^{(\theta_1-\theta_0)\mathbf{k}/2} e^{\phi_1 \mathbf{j}/2} e^{\pi \mathbf{i}/2}.$$

A straightforward computation (or a few figures) show that if $\theta_0 = \theta_1$ then $\mathfrak{F}_{\gamma_1}(t_0;t_1) = e^{(\phi_1-\phi_0)\mathbf{j}/2} e^{\pi\mathbf{i}/2} \in \mathrm{Bru}_{(13);7}$. We may keep θ_0 fixed and change θ_1 and the Bruhat cell will cycle. If $\phi_0 + \phi_1 < 0$ the cell will go through $P_{(13);7}$, $P_{(123);5}$, $P_{(13);1}$ and $P_{(123);3}$ (and back); notice that the cells $\mathrm{Bru}_{(13);7}$ and $\mathrm{Bru}_{(13);1}$ are open and correspond to intervals while the cells $\mathrm{Bru}_{(123);5}$ and $\mathrm{Bru}_{(123);3}$ have dimension 2 and correspond to transition points. If $\phi_0 + \phi_1 > 0$ the cell will go through $P_{(13);7}$, $P_{(132);6}$, $P_{(13);4}$ and $P_{(132);5}$. In the special case $\phi_0 + \phi_1 = 0$ the only transition point is $P_{e;5}$. ■

We give a few examples which will be used again later:

$$\begin{gathered}
Q_{0,1} = \Pi(e^{\pi\mathbf{j}/6}), \quad Q_{1,1} = \Pi(e^{\pi\mathbf{k}/2} e^{-\pi\mathbf{j}/12} e^{\pi\mathbf{i}/2}), \\
Q_{0,4} = \Pi(e^{\pi\mathbf{j}/12}), \quad Q_{1,4} = \Pi(e^{\pi\mathbf{k}/2} e^{-\pi\mathbf{j}/6} e^{\pi\mathbf{i}/2}), \\
Q_{0,7} = \Pi(e^{\pi\mathbf{j}/8}), \quad Q_{1,7} = \Pi(e^{-\pi\mathbf{j}/8} e^{\pi\mathbf{i}/2}).
\end{gathered}$$

We clearly have that $Q_{0,\ell}^{-1} Q_{1,\ell} \in \mathrm{Bru}_{(13);\ell}$; also, if $\mathfrak{F}_{\gamma_1}(t_i) = Q_{i,\ell}$ then γ_1 satisfies the conditions in equation 2.

A minor difficulty is that the choices of $\theta_0, \theta_1, \phi_0, \phi_1$ and of the projective transformation should be uniform. The following lemma addresses this issue.

Lemma 8.2 *Let ℓ be equal to 1, 4 or 7. Let $Q_0, Q_1 \in SO_3$ be such that*

$$Q_0^{-1} Q_1 \in \mathrm{Bru}_{(13);\ell}\,.$$

Then there exists a unique matrix $U \in Up_3^1$ such that $\Pi(Q_{0,\ell} U Q_0^{-1})(Q_i) = Q_{i,\ell}$ for $i = 0, 1$.

Proof: We have $\Pi(Q_0^{-1})(Q_0) = I$ and $\Pi(Q_0^{-1})(Q_1) = Q_0^{-1} Q_1 \in \mathrm{Bru}_{(13);\ell}$. There exists a unique $U \in Up_3^1$ with $\Pi(U)(Q_0^{-1} Q_1) = Q_{0,\ell}^{-1} Q_{1,\ell}$ and the result follows. ■

Given a curve $\gamma \in \mathcal{L}$ and a graftable arc $[t_0, t_1]$ such that $\mathfrak{F}_\gamma(t_0;t_1) \in \mathrm{Bru}_{(13);\ell}$ (where ℓ equals 1, 4 or 7) we shall assume that $\gamma_1 = \pi(Q_{0,\ell} U Q_0^{-1}) \circ \gamma$ where $U \in Up_3^1$ is as in Lemma 8.2.

Consider a curve $\gamma_1 \in \mathcal{L}$ and a graftable arc $[t_0, t_1]$ such that equation 2 is satisfied. Given a real number $s \geq 0$ we show how to perform a *graft* on the curve γ_1 around the arc $[t_0, t_1]$ to obtain a curve $\gamma_1^{[(t_0,t_1)\#s]}$. As usual, write $\tilde{\Gamma}_1 = \tilde{\mathfrak{F}}_{\gamma_1}$.

Let $\epsilon > 0$ be a small number. Define $\tilde\Gamma_1^{[(t_0,t_1)\#s]}$ by $\tilde\Gamma_1^{[(t_0,t_1)\#s]}(t) = \tilde\Gamma_1(t)$ for $t \le t_0$ and for $t \ge t_1$; otherwise

$$\tilde\Gamma_1^{[(t_0,t_1)\#s]}(t) = \begin{cases} \exp\left(\left(\frac{\theta_0}{2} + \frac{\pi(t-t_0)}{\epsilon}\right)\mathbf{k}\right) e^{\phi_0 \mathbf{j}/2}, & t_0 \le t \le t_0 + s\epsilon, \\ e^{\pi s \mathbf{k}}\tilde\Gamma_1(2t - t_0 - 2s\epsilon), & t_0 + s\epsilon \le t \le t_0 + 2s\epsilon, \\ e^{\pi s \mathbf{k}}\tilde\Gamma_1(t), & t_0 + 2s\epsilon \le t \le t_1 - 2s\epsilon, \\ e^{\pi s \mathbf{k}}\tilde\Gamma_1(2t - t_1 + 2s\epsilon), & t_1 - 2s\epsilon \le t \le t_1 - s\epsilon, \\ \exp\left(\left(\frac{\theta_1}{2} + \frac{\pi(t_1-t)}{\epsilon}\right)\mathbf{k}\right) e^{\phi_1 \mathbf{j}/2} e^{\pi \mathbf{i}/2}, & t_1 - s\epsilon \le t \le t_1. \end{cases}$$

Equation 2 guarantees compatibility and continuity. Define (of course)

$$\gamma_1^{[(t_0,t_1)\#s]}(t) = \Pi(\tilde\Gamma_1^{[(t_0,t_1)\#s]}(t))e_1.$$

A geometric description of this construction is in order: the curve γ_1 is cut at t_0 and t_1, the arc from t_0 to t_1 is rotated by an angle of $2\pi s$ around the z axis and finally arcs of circle (parallel to the plane $z = 0$) are grafted into the resulting gap (see Figure 10). Notice that $\tilde\Gamma_1^{[(t_0,t_1)\#s]}(t)$ is continuous as a function of s and t; in other words, if $\gamma_1 \in \mathcal{L}_Q$ then $s \mapsto \gamma_1^{[(t_0,t_1)\#s]}$ is a continuous path from $[0, s_{\max}]$ to $\mathcal{L}_Q$.

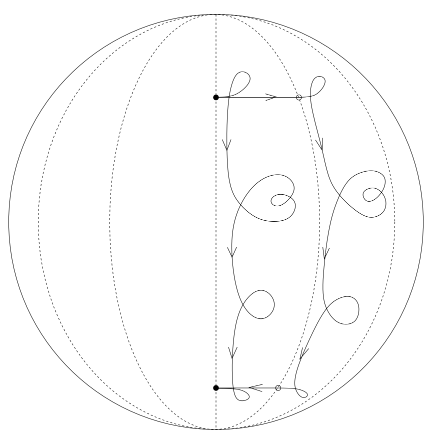

Figure 10: Grafting a curve

We now define grafting on γ provided $\mathfrak{F}_\gamma(t_0; t_1) \in \mathrm{Bru}_{(13);\ell}$ where ℓ equals 1, 4 or 7. Let U be as in Lemma 8.2 and $M = Q_{0,\ell}UQ_0^{-1} \in SL_3$ so that the projective transformation $\pi(M)$ takes γ to $\gamma_1 = \pi(M)\circ\gamma$ satisfying $\mathfrak{F}_{\gamma_1}(t_i) = Q_{i,\ell}$. Next graft γ_1 as above to obtain $\gamma_1^{[(t_0,t_1)\#s]}$ and define $\gamma^{[(t_0,t_1)\#s]} = \pi(M^{-1})\circ\gamma_1^{[(t_0,t_1)\#s]}$. Define a group homomorphism $A : \mathbb{R} \to SL_3$ by $A(s) = M^{-1}\Pi(e^{s\pi\mathbf{k}})M = \exp(sa)$,

$a \in sl_3$; notice that $A(1) = \exp(a) = I$. We have

$$\gamma^{[(t_0,t_1)\#s]}(t) = \begin{cases} \gamma(t), & t \leq t_0, \\ \pi(\exp(sa))\gamma(t), & t_0 + 2s\epsilon \leq t \leq t_1 - 2s\epsilon, \\ \gamma(t), & t \geq t_1. \end{cases}$$

If s is a positive integer, the curves $\gamma^{[(t_0,t_1)\#s]}$ (grafting) and $\gamma^{[t_0\#s;t_1\#s]}$ (adding loops, as in Section 6, just before Lemma 6.1) are very similar. The only significant difference is that the "loops" in the first curve are closed convex curves which are not quite circles. Since the space of closed convex curves with a given base point is contractible these loops can easily be made round.

We write $A \Subset B$ if the closure of A is contained in the interior of B. Notice that if A is open and closed then $A \Subset A$.

Lemma 8.3 *Let ℓ equal 1, 4 or 7. Let K be a compact manifold and $f : K \to \mathcal{L}$ be a continuous map. Let $K_0 \subset K$ be a compact subset. Assume that there exist continuous functions $t_0 < t_1 : K_0 \to (0,1)$ such that, for all $p \in K_0$, $\mathfrak{F}_{f(p)}(t_0;t_1) \in \mathrm{Bru}_{(13);\ell}$. Let $W_0 \Subset K_0$ be an open set. Then, for sufficiently small $\epsilon > 0$, there exist a homotopy $H : [0,1] \times K \to \mathcal{L}$ and a function $A : [0,1] \times K \to SL_3$ with the following properties:*

(a) $t_0(p) + 8\epsilon < t_1(p) - 8\epsilon$ *for all* $p \in K_0$;

(b) $A(p) = I$ *for all* $p \in (K \smallsetminus K_0) \cup W_0$;

(c) $H(0,p) = f(p)$ *for all* $p \in K$;

(d) $H(s,p) = f(p)$ *for all* $p \in K \smallsetminus K_0$ *and all* $s \in [0,1]$;

(e) $H(s,p)(t) = f(p)(t)$ *for all* $p \in K_0$ *and for all* $t \notin (t_0(p), t_1(p))$ *and all* $s \in [0,1]$;

(f) $H(s,p)(t) = \pi(A(s,p))(f(p)(t))$ *for all* $p \in K_0$ *and for all* $t \in (t_0(p) + 8\epsilon, t_1(p) - 8\epsilon)$ *and all* $s \in [0,1]$;

(g) $H(1,p) = f(p)^{[(t_0(p)+4\epsilon)\#2;(t_1(p)-4\epsilon)\#2]}$ *for all* $p \in W_0$.

Proof: Let $W_1 \Subset K_0$ be an open set such that $W_0 \Subset W_1$. Let $u : K \to [0,1]$ be a smooth function with $u(p) = 0$ for $p \notin K_0$ and $u(p) = 1$ for $p \in W_1$.

For $s \in [0,1/2]$, define $H(s,p)$ by grafting $f(p)$:

$$H(s,p) = f(p)^{[(t_0(p),t_1(p))\#(4su(p))]}.$$

Notice that for $p \in W_1$ we have $H(1/2,p) = f(p)^{[(t_0(p),t_1(p))\#2]}$. For $s \in [1/2,1]$ and $p \notin W_1$ we define $H(s,p) = H(1/2,p)$. For $p \in W_1$ we use the interval to round up the loops introduced by grafting. The margin $W_1 \smallsetminus W_0$ is needed for compatibility but for $p \in W_0$ we may assume that $H(1,p) = f(p)^{[(t_0(p)+4\epsilon)\#2;(t_1(p)-4\epsilon)\#2]}$, completing our proof. ■

We are now ready to prove the easier cases of our main theorem; these are also proved in [20] using different ideas.

Corollary 8.4 *Let $z \in \mathbb{S}^3$ with $\Pi(z) \in \mathrm{Bru}_{(13);1} \cup \mathrm{Bru}_{(13);4} \cup \mathrm{Bru}_{(13);7}$. Then the inclusion $\mathcal{L}_z \subset \mathcal{I}_z$ is a weak homotopy equivalence.*

Since these spaces are Hilbert manifolds, they are actually diffeomorphic ([3]).

Proof: Let ℓ be such that $\Pi(z) \in \mathrm{Bru}_{(13);\ell}$. Let K be a compact manifold and $f : K \to \mathcal{L}_z$ be a continuous map. For sufficiently small $\epsilon > 0$ we have $\mathfrak{F}_{f(p)}(\epsilon, 1-\epsilon) \in \mathrm{Bru}_{(13);\ell}$. Apply Lemma 8.3 to f with $t_0 = \epsilon$, $t_1 = 1-\epsilon$, $K_0 = W_0 = K$ to deduce that f is homotopic to $f^{[t_a\#2;t_b\#2]}$ and therefore loose. It now follows from Proposition 6.4 that f is homotopic to a constant in $\mathcal{L}_z$ if and only if f is homotopic to a constant in $\mathcal{I}_z$. Together with Lemma 6.3, this completes the proof. ■

Part of Little's Theorem is that the set $\mathcal{L}_{-\mathbf{1},c}$ of simple locally convex curves is a connected component of $\mathcal{L}_{-\mathbf{1}}$: Fenchel proves that simple closed locally convex curves are convex ([8]; see also [15] and [25]); we often use this fact. The other part of Little's Theorem is that, once convex curves have been removed, the sets $\mathcal{L}_{+\mathbf{1}}$ and $\mathcal{L}_{-\mathbf{1},n}$ are path connected. This is again a corollary of Lemma 8.3.

Corollary 8.5 *The sets $\mathcal{L}_{+\mathbf{1}}$ and $\mathcal{L}_{-\mathbf{1},n}$ are path connected.*

Proof: Consider a map from $\mathbb{S}^0$ (two points) to either of these spaces (i.e., two curves): the map is homotopic to a constant in $\mathcal{I}_{\pm\mathbf{1}}$. Each of the two curves γ_0 and γ_1 may be assumed generic and therefore to have a transversal self-intersection. As in figure 11, near the self-intersection there exist t_0 and t_1 such that $\mathfrak{F}_\gamma(t_0; t_1) \in \mathrm{Bru}_{(13);7}$ (there are other pairs for which this expression belongs to any of the other three open cells). ■

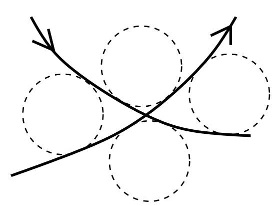

Figure 11: A transversal self-intersection

9 Good and bad steps

Given a locally convex curve $\gamma : [t_0, t_1] \to \mathbb{S}^2$, we now define the *next step* function $\mathrm{ns}_\gamma : [t_0, t_1^-] \to [t_0, t_1]$, or, to make it shorter when γ is clear from the context,

$t^+ = \mathrm{ns}_\gamma(t)$ (the number $t_1^- \in [t_0, t_1)$ will also be defined). Given $t \in [t_0, t_1]$ let t^+ be the smallest $\tilde{t} > t$ such that $\Gamma(t; \tilde{t}) \notin \mathrm{Bru}_{(13),2}$; if no such $\tilde{t}$ exists then t^+ is undefined. Since $\mathrm{Bru}_{(13),2}$ is an open set, ns_γ is a continuous function (where defined). Also, the function ns_γ is strictly increasing with continuous inverse, which will be denoted by $t^- = \mathrm{ns}_\gamma^{-1}(t)$. Unless $\gamma : [t_0, t_1] \to \mathbb{S}^2$ is convex, ns_γ is a strictly increasing continuous homeomorphism from $[t_0, t_1^-]$ to $[t_0^+, t_1]$. On the other hand, ns_γ is usually not differentiable (even if γ is smooth).

Geometrically speaking, t_0^+ is the point at which the arc $\gamma|_{[t_0, t_0^+]}$ is still convex but about to somehow lose convexity. This can occur in five different ways corresponding to five Bruhat cells to which $\Gamma(t_0; t_0^+)$ may belong. The two generic cases are when γ is about to leave the hemisphere defined by $\Gamma(t_0)$ (but not at the point $\gamma(t_0)$) or, conversely, when the geodesic defined by $\Gamma(t)$ passes through $\gamma(t_0)$ (but not aligned with $\gamma'(t_0)$) so that γ is about to enter its own convex hull: these correspond to $P_{(123);6}$ and $P_{(132);0}$, in this order, and to the first two diagrams in Figure 5: Notice that these matrices have two inversions and therefore their Bruhat cells have dimension 2. Two more exceptional cases correspond to the matrices $P_{(23);2}$ and $P_{(12);4}$ which, having one inversion, correspond to Bruhat cells of dimension 1. The two cases correspond to the third and fourth diagram in Figure 5: the curve may self-intersect by coming back to $\gamma(t_0)$ (but with non-aligned tangent vectors) or it may tangentially touch the geodesic defined by $\Gamma(t_0)$ (but not at $\gamma(t_0)$). The fifth and most exceptional case corresponds to $P_{e;0} = I$, with Bruhat cell of dimension 0: the curve may come back to $\gamma(t_0)$ with tangent vectors also aligned (as in the fifth diagram).

We are particularly interested in this fifth and most degenerate case. A *step* (for γ) is an interval $[t_0, t_1]$ with $t_1 = t_0^+$. A step is *bad* if $\Gamma(t_1) = \Gamma(t_0)$ (this is the fifth case) and *good* otherwise. Notice that the set of good steps is open (in the space of steps). A curve γ is multiconvex if and only if, for $t_0 = 0$, the steps $[t_0, t_1], [t_1, t_2], \ldots$ are all bad (here $t_j = \mathrm{ns}_\gamma^j(t_0)$). Conversely, if a curve is complicated (i.e., not multiconvex) then, again with $t_j = \mathrm{ns}_\gamma^j(t_0)$, there exists a good step $[t_j, t_{j+1}] \subset [0, 1]$

A *good arc* for a locally convex curve γ is an interval $[t_0, t_1] \subset (0, 1)$ such that:

(a) $t_0 < t_1^- < t_0^+ < t_1$;

(b) if $t \in [t_0, t_1^-]$ then $[t, t^+]$ is a good step;

(c) if $t_0 \le t_a < t_a^+ < t_b \le t_1$ then $\mathfrak{F}_\gamma(t_a; t_b)$ is in one of the following Bruhat cells: $\mathrm{Bru}_{(13);1}$, $\mathrm{Bru}_{(13);4}$, $\mathrm{Bru}_{(13);7}$, $\mathrm{Bru}_{(123);3}$, $\mathrm{Bru}_{(123);5}$, $\mathrm{Bru}_{(132);5}$ or $\mathrm{Bru}_{(132);6}$.

The first three matrices in item (c) correspond of course to open cells as in Figure 4 above; the last four are shown in Figure 12.

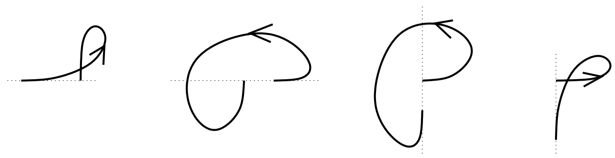

Figure 12: Good arcs

A *good pair of arcs* for a locally convex curve γ consists of two good arcs $[t_0, t_1] \subset [\tilde{t}_0, \tilde{t}_1] \subset (0,1)$ such that $\tilde{t}_0 < t_0 < t_1 < \tilde{t}_1$ and if $\mathfrak{F}_\gamma(t_0; t_1) \notin \mathrm{Bru}_{(13),1} \cup \mathrm{Bru}_{(13),4}$ then $\mathfrak{F}_\gamma(\tilde{t}_0; \tilde{t}_1) \in \mathrm{Bru}_{(13),7}$.

Lemma 9.1 *Let K be a compact manifold; let $f : K \to \mathcal{L}$ be a family of locally convex curves. Let $t_0, t_1 : K \to (0,1)$ be continuous functions with $\mathrm{ns}_{f(p)}(t_0(p)) = t_1(p)$ such that $[t_0(p), t_1(p)]$ is always a good step.*

(a) There exists $\epsilon > 0$ such that $[t_0(p) - \epsilon, t_1(p) + \epsilon] \subset (0,1)$ is always a good arc.

(b) For any $\epsilon > 0$ there exist continuous functions $\tilde{t}_0 < \hat{t}_0 < \hat{t}_1 < \tilde{t}_1 : K \to (0,1)$ with $t_0 - \epsilon < \tilde{t}_0 < \hat{t}_0 < t_0$ and $t_1 < \hat{t}_1 < \tilde{t}_1 < t_1 + \epsilon$ such that, for all $p \in K$, $[\hat{t}_0(p), \hat{t}_1(p)] \subset [\tilde{t}_0(p), \tilde{t}_1(p)]$ is a good pair of arcs for $f(p)$.

Proof: Consider the union of all allowed cells in the definition of a good arc for $\mathfrak{F}_\gamma(t_a; t_b)$ if $t_a < t_a^+ < t_b$:

$$\begin{aligned} A_1 = {} & \mathrm{Bru}_{(13);1} \cup \mathrm{Bru}_{(13);4} \cup \mathrm{Bru}_{(13);7} \cup \\ & \cup \mathrm{Bru}_{(123);3} \cup \mathrm{Bru}_{(123);5} \cup \mathrm{Bru}_{(132);5} \cup \mathrm{Bru}_{(132);6} \subset SO_3. \end{aligned}$$

Consider also the set of all allowed cells for $\mathfrak{F}_\gamma(t_a; t_b)$ if $t_a < t_b$:

$$A_2 = A_1 \cup \mathrm{Bru}_{(13);2} \cup \mathrm{Bru}_{(123);6} \cup \mathrm{Bru}_{(132);0} \cup \mathrm{Bru}_{(23);2} \cup \mathrm{Bru}_{(12);4} \subset SO_3.$$

We claim that the sets A_1 and A_2 are both open. For A_1, each of the four 2-cells included is sandwiched between two of the included 3-cells: $\mathrm{Bru}_{(123);3}$ between $\mathrm{Bru}_{(13);1}$ and $\mathrm{Bru}_{(13);7}$; $\mathrm{Bru}_{(123);5}$ between $\mathrm{Bru}_{(13);1}$ and $\mathrm{Bru}_{(13);7}$; $\mathrm{Bru}_{(132);5}$ between $\mathrm{Bru}_{(13);4}$ and $\mathrm{Bru}_{(13);7}$; $\mathrm{Bru}_{(132);6}$ between $\mathrm{Bru}_{(13);4}$ and $\mathrm{Bru}_{(13);7}$. For A_2, since all four 3-dimensional cells are included we need only check that each of the two 1-dimensional cells included is surrounded by 2- and 3-dimensional cells which are also included: $\mathrm{Bru}_{(23);1}$ is surrounded by the four 3-dimensional cells plus $\mathrm{Bru}_{(132);0}$, $\mathrm{Bru}_{(123);6}$, $\mathrm{Bru}_{(123);3}$ and $\mathrm{Bru}_{(132);6}$; $\mathrm{Bru}_{(12);4}$ is surrounded by the four 3-dimensional cells plus $\mathrm{Bru}_{(123);6}$, $\mathrm{Bru}_{(132);0}$, $\mathrm{Bru}_{(132);5}$ and $\mathrm{Bru}_{(123);5}$. This completes the proof of the claim

By hypothesis, if $t_0(p) \le t_a < t_b \le t_1(p)$ then $\mathfrak{F}_{f(p)}(t_a; t_b) \in A_2$. Thus, using the compactness of K and the fact that A_2 is open, for sufficiently small ϵ_2, if

$t_0(p) - \epsilon_2 < t_a < t_b < t_1(p) + \epsilon_2$ then $\mathfrak{F}_{f(p)}(t_a; t_b) \in A_2$. In particular, $[t_a, t_a^+]$ is a good step. For any good step $[t_a, t_a^+]$ for a locally convex curve γ there exists $\epsilon_a > 0$ such that if $t_a^+ < t_b < t_a^+ + \epsilon_a$ then $\mathfrak{F}_\gamma(t_a; t_b) \in A_1$. Again by compactness, there exists therefore $\epsilon \in (0, \epsilon_2)$ such that, for all p, if $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$ then $\mathfrak{F}_{f(p)}(t_a; t_b) \in A_1$, proving (a).

For item (b), assume arcs parametrized by a constant multiple (depending on p only) of arc length. We claim that for sufficiently small $\epsilon_b > 0$, we may take $\tilde{t}_0(p) = t_0(p) - 2\epsilon_b$, $\hat{t}_0(p) = t_0(p) - \epsilon_b$, $\hat{t}_1(p) = t_1(p) + \epsilon_b$, $\tilde{t}_1(p) = t_1(p) + 2\epsilon_b$: $[\hat{t}_0(p), \hat{t}_1(p)] \subset [\tilde{t}_0(p), \tilde{t}_1(p)]$ is a good pair of arcs for $f(p)$.

Let $B = \mathrm{Bru}_{(123);6} \cup \mathrm{Bru}_{(132);0} \cup \mathrm{Bru}_{(23);2} \cup \mathrm{Bru}_{(12);4} \subset A_2 \subset SO_3$; the set B is a topological manifold of dimension 2 homeomorphic to $\mathbb{S}^1 \times (0,1)$; the subsets $\mathrm{Bru}_{(23);2}, \mathrm{Bru}_{(12);4} \subset B$ are closed with disjoint neighborhoods $B_2, B_4 \subset B$, respectively. We may assume the closures of B_2 and B_4 in B to be disjoint. Let $s : K \to B$ be defined by $s(p) = \mathfrak{F}_{f(p)}(t_0(p); t_1(p))$; let $U_2 = s^{-1}(B_2)$, $U_4 = s^{-1}(B_4)$. Define open sets $U_6 \Subset s^{-1}(\mathrm{Bru}_{(123);6})$ and $U_0 \Subset s^{-1}(\mathrm{Bru}_{(132);0})$, so that the sets U_i form an open cover of K. For sufficiently small $\epsilon > 0$, if $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$ and $p \in U_6$ (resp. $p \in U_0$) then $\mathfrak{F}_{f(p)}(t_a; t_b) \in \mathrm{Bru}_{(13);4}$ (resp. $\mathfrak{F}_{f(p)}(t_a; t_b) \in \mathrm{Bru}_{(13);1}$). For $p \in U_6 \cup U_0$, therefore, pairs of arcs will be good. We must focus on $p \in U_2$ and $p \in U_4$.

Assume $p \in U_2$. For small $\epsilon > 0$, we may assume that the arc $[t_0(p) - \epsilon, t_1(p) + \epsilon]$ has at most one self intersection. Let $V_2 \subset U_2$ be the open set of points $p \in U_2 \subset K$ for which there exist $t_c, t_d \in (t_0(p) - \epsilon, t_1(p) + \epsilon)$ with $t_c < t_d$, $f(p)(t_c) = f(p)(t_d)$ so that $t_c^+ = t_d$ and $\mathfrak{F}_{f(p)}(t_c; t_d) \in \mathrm{Bru}_{(23);2}$. By taking $\epsilon > 0$ sufficiently small, we may assume that if $p \in s^{-1}(\mathrm{Bru}_{(123);6}) \smallsetminus V_2$ and $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$ then $\mathfrak{F}_{f(p)}(t_a; t_b) \in \mathrm{Bru}_{(13);4}$. Similarly, we may assume that if $p \in s^{-1}(\mathrm{Bru}_{(132);0}) \smallsetminus V_2$ and $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$ then $\mathfrak{F}_{f(p)}(t_a; t_b) \in \mathrm{Bru}_{(13);1}$.

If $p \in V_2$ and $t \geq t_c$ set $h_+(t) = t^+$; if $t < t_c$, let $\tilde{t} = h_+(t) \in (t_d, t_1 + \epsilon)$ be uniquely defined by $\mathfrak{F}_{f(p)}(t; \tilde{t}) \in \mathrm{Bru}_{(123);3}$; more geometrically, draw a tangent geodesic to γ at t: the curve γ intersects the geodesic at $\tilde{t}$. The function $h_+ : [t_c - \delta, t_c + \delta] \to [t_d, t_d + \tilde{\delta}]$ is continuous, decreasing for $t < t_c$ and increasing for $t > t_c$ and satisfies $h_+(t_c) = t_d$ and $h'_+(t_c) = 0$. Similarly, if $t \leq t_d$ let $h_-(t) = t^-$; otherwise, if $t > t_d$, let $\tilde{t} = h_-(t) \in (t_0 - \epsilon, t_c)$ be defined by $\mathfrak{F}_{f(p)}(\tilde{t}; t) \in \mathrm{Bru}_{(132);6}$; again, the function $h_- : [t_d - \delta, t_d + \delta] \to [t_c - \tilde{\delta}, t_c]$ is continuous, increasing for $t < t_d$ and decreasing for $t > t_d$ and satisfies $h_-(t_d) = t_c$ and $h'_-(t_d) = 0$ (see Figure 13). By taking ϵ small we may assume that the functions h_+ and h_- are always $(1/2)$-Lipschitz, i.e., that $|h_+(t_a) - h_+(\tilde{t}_a)| \leq |t_a - \tilde{t}_a|/2$ and $|h_-(t_b) - h_-(\tilde{t}_b)| \leq |t_b - \tilde{t}_b|/2$.

If $p \in V_2$, $t_0 - \epsilon < t_a < t_a^+ < t_b < t_1 + \epsilon$ and $t_a > h_-(t_b)$ then $\mathfrak{F}_\gamma(t_a; t_b) \in \mathrm{Bru}_{(13),4}$. Similarly, if $t_b < h_+(t_a)$ then $\mathfrak{F}_\gamma(t_a; t_b) \in \mathrm{Bru}_{(13),1}$. Thus, if $\mathfrak{F}_\gamma(t_a; t_b) \notin \mathrm{Bru}_{(13),1} \cup \mathrm{Bru}_{(13),4}$ then $t_a \leq h_-(t_b) \leq t_c < t_d \leq h_+(t_a) \leq t_b$. Conversely, if

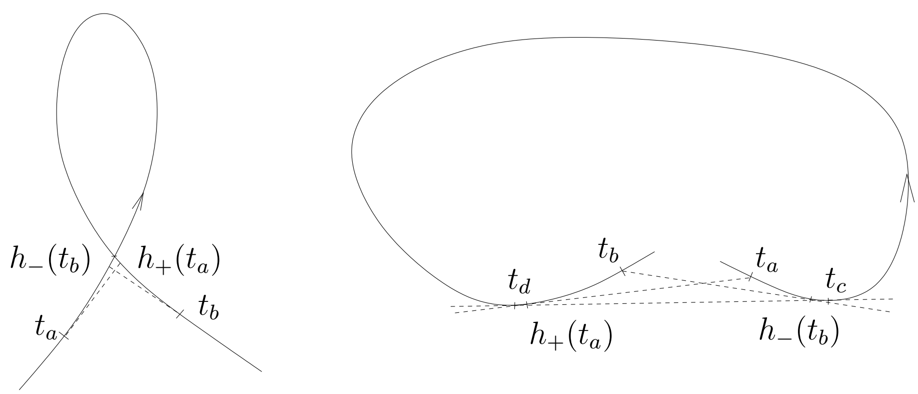

Figure 13: The functions h_+ and h_-

$t_a < h_-(t_b) < t_c < t_d < h_+(t_a) < t_b$ then $\mathfrak{F}_\gamma(t_a; t_b) \in \mathrm{Bru}_{(13),7}$. But the Lipschitz condition means that if $t_0 - \epsilon < t_a - \epsilon_b < t_a < t_a^+ < t_b < t_b + \epsilon_b < t_1 + \epsilon$ and $\mathfrak{F}_\gamma(t_a; t_b) \notin \mathrm{Bru}_{(13),1} \cup \mathrm{Bru}_{(13),4}$ then $\mathfrak{F}_\gamma(t_a - \epsilon_b; t_b + \epsilon_b) \in \mathrm{Bru}_{(13),7}$.

A similar construction holds in U_4, only the Bruhat classes change and the geometric construction of $h_\pm$ is different (see Figure 13): to define $h_+(t_a)$ we construct a tangent line to γ which passes through t_a. Alternatively, we apply Arnold duality ([2], [20]) to go from the U_4 to the U_2 scenario. ■

10 Complicated curves

Recall that a curve $\gamma \in \mathcal{L}_I$ is *complicated* if it belongs to

$$\mathcal{Y}_Q = \mathcal{L}_Q \smallsetminus \bigcup_k \mathcal{M}_k.$$

Lemma 10.1 *Let K be a compact manifold and let $f : K \to \mathcal{L}_Q$ be a continuous map. If the image of f is contained in $\mathcal{Y}_Q$ then f is loose.*

Proof: Let $\tilde{U}_0 \subseteq K$ be the open set of elements $p \in K$ for which $[0, 0^+]$ is a good step. More generally, let $\tilde{U}_j \subseteq K$ be the open set of elements $p \in K$ for which $\mathrm{ns}_{f(p)}^{j+1}(0) < 1$ and $[\mathrm{ns}_{f(p)}^{j}(0), \mathrm{ns}_{f(p)}^{j+1}(0)]$ is a good step. Since all curves $f(p)$ are complicated, the sets $\tilde{U}_j$ cover K. By compactness of K, there exists an integer J such that $K = \bigcup_{j<J} \tilde{U}_j$. Define compact sets $K_j \subset \tilde{U}_j$ such that the interiors U_j of K_j already cover K.

Again by compactness, there exists $\epsilon_{J-1} > 0$ such that if $p \in K_{J-1}$ and $|t_{J-1} - \mathrm{ns}_{f(p)}^{J-1}(0)| < \epsilon_{J-1}$ then $\mathrm{ns}_{f(p)}(t_{J-1}) < 1$ and $[t_{J-1}, \mathrm{ns}_{f(p)}(t_{J-1})]$ is a good step (for $f(p)$). Define the continuous function $t_{J-1} : K_{J-1} \to [0, 1]$ by $t_{J-1}(p) = \epsilon_{J-1}/2 + \mathrm{ns}_{f(p)}^{J-1}(0)$ Similarly, there exists $\epsilon_{J-2} > 0$ such that if $p \in K_{J-2}$ and

$|t_{J-2} - \mathrm{ns}^{J-2}_{f(p)}(0)| < \epsilon_{J-2}$ then $\mathrm{ns}_{f(p)}(t_{J-2}) < 1$, $\mathrm{ns}_{f(p)}(t_{J-2}) < t_{J-1}(p)$ if $p \in K_{J-1}$ and $[t_{J-2}, \mathrm{ns}_{f(p)}(t_{J-2})]$ is a good step. Proceed in this manner so that we have continuous functions $t_j : K_j \to (0,1)$, $0 \le j < J$, such that if $p \in K_j$ then $[t_j(p), \mathrm{ns}_{f(p)}(t_j(p))] \subset (0,1)$ is a good step for $f(p)$ and if $p \in K_{j_1} \cap K_{j_2}$, $j_1 < j_2$, then $\mathrm{ns}_{f(p)}(t_{j_1}(p)) < t_{j_2}(p)$. Again by compactness, there exists $\epsilon > 0$ such that $t_j(p) > \epsilon$, $\mathrm{ns}_{f(p)} < 1-\epsilon$ and if $p \in K_{j_1} \cap K_{j_2}$, $j_1 < j_2$, then $\mathrm{ns}_{f(p)}(t_{j_1}(p)) + \epsilon < t_{j_2}(p)$.

Thus if $p \in K_j$ then the interior of the interval $\hat{I}_j = [t_j(p) - \epsilon/3, \mathrm{ns}_{f(p)}(t_j(p)) + \epsilon/3]$ contains the good step $[t_j(p), \mathrm{ns}_{f(p)}(t_j(p))]$. Notice furthermore that for given p the intervals $\hat{I}_j$ are disjoint. From Lemma 9.1 we can now define functions $a_j < b_j < c_j < d_j : K \to (0,1)$ such that, for all $p \in K$, $a_j(p), b_j(p), c_j(p), d_j(p) \in \hat{I}_j(p)$ and, for $p \in K_j$, $[b_j(p), c_j(p)] \subset [a_j(p), d_j(p)]$ is a good pair of arcs. Define $I_j(p) = [b_j(p), c_j(p)]$ and $\tilde{I}_j(p) = [a_j(p), d_j(p)]$.

The strategy now is to deform curves in each interval I_j independently. More precisely, define

$$\begin{aligned} U_{j,1} &= \{p \in K_j; \mathfrak{F}_{f(p)}(b_j(p); c_j(p)) \in \mathrm{Bru}_{(13),1}\}, \\ U_{j,4} &= \{p \in K_j; \mathfrak{F}_{f(p)}(b_j(p); c_j(p)) \in \mathrm{Bru}_{(13),4}\}, \\ U_{j,7} &= \{p \in K_j; \mathfrak{F}_{f(p)}(a_j(p); d_j(p)) \in \mathrm{Bru}_{(13),7}\}. \end{aligned}$$

By definition of good pair of arcs, $U_j = U_{j,1} \cup U_{j,4} \cup U_{j,7}$. Consider open sets $W_{j,\ell}$ and compact sets $K_{j,\ell}$ such that $W_{j,\ell} \Subset K_{j,\ell} \Subset U_{j,\ell}$ and such that the sets $W_{j,\ell}$ cover K.

Let $f_0 = f$; apply Lemma 8.3 (with $\ell = 1$, $K_0 = K_{0,1}$, $W_0 = W_{0,1}$, $t_0 = b_0$ and $t_1 = c_0$) to define a homotopy from f_0 to another function $f_{0,1}$ with $f_{0,1}(p) = f_0(p)^{[b_0\#2; c_0\#2]}$ for all $p \in W_{0,1}$. Apply the same lemma (now with $\ell = 4$, $K_0 = K_{0,4}$, $W_0 = W_{0,4}$, $t_0 = b_1$ and $t_1 = c_1$) to define a homotopy from $f_{0,1}$ to $f_{0,4}$ with $f_{0,4}(p) = f_{0,1}(p)^{[b_0\#2; c_0\#2]}$ for all $p \in W_{0,4}$; notice that since $K_{0,1}$ and $K_{0,4}$ are disjoint the two constructions do not interfere with one another. Apply Lemma 8.3 yet another time (now with $\ell = 7$, $K_0 = K_{0,7}$, $W_0 = W_{0,7}$, $t_0 = a_1$ and $t_1 = d_1$) to define a homotopy from $f_{0,4}$ to $f_{0,7}$ with $f_{0,7}(p) = f_{0,4}(p)^{[a_0\#2; d_0\#2]}$ for all $p \in W_{0,7}$; notice that $\mathfrak{F}_{f_{0,4}(p)}(a_0(p)) = \mathfrak{F}_{f(p)}(a_0(p))$ and $\mathfrak{F}_{f_{0,4}(p)}(d_0(p)) = \mathfrak{F}_{f(p)}(d_0(p))$. Even though the interval (a_0, d_0) contains the points b_0 and c_0, the loops created in the two first steps were not destroyed but merely pushed around by a projective transformation. We may therefore define a homotopy from $f_{0,7}$ to f_1 such that if $p \in W_{0,1} \cup W_{0,4}$ then $f_1(p) = \gamma^{[b_0\#2; c_0\#2]}$ for some $\gamma \in \mathcal{L}_Q$; if $p \in W_{0,7}$ then $f_1(p) = \gamma^{[a_0\#2; d_0\#2]}$ for some $\gamma \in \mathcal{L}_Q$.

Repeat the process to define a homotopy from f_1 to f_2 and so on until f_J such that, finally, for f_J we have:

- if $p \in W_{j,1} \cup W_{j,4}$ then $f_J(p) = \gamma^{[b_j\#2; c_j\#2]}$ for some $\gamma \in \mathcal{L}_Q$;
- if $p \in W_{j,7}$ then $f_J(p) = \gamma^{[a_j\#2; d_j\#2]}$ for some $\gamma \in \mathcal{L}_Q$.

Since the sets $W_{j,\ell}$ cover K we have from Lemma 6.6 that f_J is loose. ■

We now prove that the inclusions $\mathcal{Y}_z \subset \mathcal{I}_z$ are weak homotopy equivalences.

Proof of Proposition 1.4: Let K be a compact manifold of dimension n and let $f : K \to \mathcal{I}_z$ a continuous map. From Lemma 6.3, for sufficiently large m, f is homotopic to $f_1 = F_{2m} \circ f$ and the image of f_1 is contained in $\mathcal{L}_z$. Furthermore, the total curvature of $f_1(p) = (F_{2m} \circ f)(p)$ tends to infinity when m tends to infinity. We may therefore choose m large enough so that $\operatorname{tot}(f_1(p)) > 8(n+1)\pi$ for all $p \in K$. In particular, the image of f_1 is disjoint from $\mathcal{M}_k$ for all $k \leq n+1$. For $k > n+1$, the codimension of $\mathcal{M}_k$ equals $2k-2 > 2n \geq n$: by transversality, we may perturb f_1 to define a homotopic map $f_2 : K \to \mathcal{L}_z$ whose image is disjoint from all submanifolds $\mathcal{M}_k$, so that the image of f_2 is contained in $\mathcal{Y}_z$. In particular the maps $i_* = \pi_n(i) : \pi_n(\mathcal{Y}_z) \to \pi_n(\mathcal{I}_z)$ are surjective.

Conversely, let B be a compact manifold of dimension $n+1$ with boundary $K = \partial B$. Let $g : B \to \mathcal{I}_z$ be a continuous map with the image of $f = g|_K$ contained in $\mathcal{Y}_z$. We prove that there exists a map $\tilde{g} : B \to \mathcal{Y}_z$ with $\tilde{g}|_K = f$. Indeed, let $g_1 = F_{2m} \circ g : B \to \mathcal{I}_z$. Again from Lemma 6.3, for sufficiently large m we have that the image of g_1 is contained in $\mathcal{L}_z$. From Lemma 10.1, $f : K \to \mathcal{L}_z$ is homotopic (in $\mathcal{L}_z$) to $f_1 = g_1|_K$. We therefore obtain a map $g_2 : B \to \mathcal{L}_z$ with $g_2|_K = f$. By Lemma 7.5, there exists $g_3 : B \to \mathcal{Y}_z \subset \mathcal{L}_z$ with $g_3|_K = f$. In particular the maps $i_* = \pi_n(i) : \pi_n(\mathcal{Y}_z) \to \pi_n(\mathcal{I}_z)$ are injective. ■

11 Proof of Theorems 1 and 2 and final remarks

We now have all the tools required to complete the proof of Theorem 2; Theorem 1 is then a special case.

Proof of Theorem 2: Recall that $\mathcal{M}_k$ is either empty or a contractible submanifold of codimension $2k-2$. As in the proof of Lemma 7.5, for each k, if $\mathcal{M}_k \neq \emptyset$ let $V_k \subset \mathcal{L}_z$ be a closed tubular neighborhood of $\mathcal{M}_k$ with interior $U_k \subset V_k$. If $\mathcal{M}_k = \emptyset$, let $U_k = V_k = \emptyset$; also, for $k = 1$ if $\mathcal{M}_k \neq \emptyset$ then $\mathcal{M}_k$ is a contractible connected component and $U_k = V_k = \mathcal{M}_k$. As before, assume the sets V_k to be disjoint. If $\mathcal{M}_k \neq \emptyset$, let $\psi_k = (\psi_{k,1}, \psi_{k,2}) : V_k \to \mathcal{H} \times \mathbb{D}^{2k-2}$ be a homeomorphism ($\mathcal{H}$ is the Hilbert space). Again, assume that $\mathcal{M}_k = \psi_{k,2}^{-1}(\{0\})$. Assume furthermore that $D_k = h_{2k-2}^{-1}(V_k) \subset \mathbb{S}^{2k-2}$ is a closed disk with smooth boundary. Assume also that $\psi_{k,1} \circ h_{2k-2}$ is constant equal to 0 in D_k and that h_{2k-2} is a homeomorphism from D_k to $\mathcal{D}_k = h_{2k-2}(D_k) = \psi_{k,1}^{-1}(\{0\}) \subset V_k$. Let $\tilde{\mathcal{Y}}_z \subset \mathcal{Y}_z$ and $\tilde{\mathcal{L}}_z \subset \mathcal{L}_z$ be defined by

$$\tilde{\mathcal{Y}}_z = \mathcal{L}_z \smallsetminus \bigcup_{\mathcal{M}_k \neq \emptyset} U_k, \quad \tilde{\mathcal{L}}_z = \tilde{\mathcal{Y}}_z \cup \bigcup_{\mathcal{M}_k \neq \emptyset} \mathcal{D}_k.$$

Since $\mathcal{H} \times \mathbb{S}^{2k-3} \subset \mathcal{H} \times (\mathbb{D}^{2k-2} \smallsetminus \{0\})$ is a deformation retract it follows that so is $\tilde{\mathcal{Y}}_z \subset \mathcal{Y}_z$. It thus follows from Proposition 1.4 that $\tilde{\mathcal{Y}}_z \approx \mathcal{I}_z \approx \Omega\mathbb{S}^3$. Similarly, since $(\mathcal{H} \times \mathbb{S}^{2k-3}) \cup (\{0\} \times \mathbb{D}^{2k-2}) \subset \mathcal{H} \times \mathbb{D}^{2k-2}$ is a deformation retract, so is $\tilde{\mathcal{L}}_z \subset \mathcal{L}_z$.

If neither z nor $-z$ is convex, $\mathcal{M}_k = \emptyset$ for all k and $\tilde{\mathcal{Y}}_z = \tilde{\mathcal{L}}_z$ and we are done. If z (resp. $-z$) is convex then $\mathcal{M}_k \neq \emptyset$ for k odd (resp. even). Thus, for z convex, $\tilde{\mathcal{L}}_z$ is obtained from $\tilde{\mathcal{Y}}_z \approx \Omega\mathbb{S}^3$ by gluing disks $\mathbb{D}^0$ (i.e., adding a contractible connected component), $\mathbb{D}^4$, $\mathbb{D}^8$, ... Similarly, for $-z$ convex, $\tilde{\mathcal{L}}_z$ is obtained from $\tilde{\mathcal{Y}}_z \approx \Omega\mathbb{S}^3$ by gluing disks $\mathbb{D}^2$, $\mathbb{D}^6$, $\mathbb{D}^{10}$, ... The maps h_{2k-2} guarantee that the spheres along which these disks are being glued are nullhomotopic in $\tilde{\mathcal{Y}}_z$ and therefore gluing a disk is (homotopically) equivalent to gluing a sphere: the theorem follows. ■

The question of how the spaces $\mathcal{L}_z$ fit together still requires some clarification. It should be noted that the map from $\mathcal{L}$ to $\mathbb{S}^3$ taking γ to $\tilde{\mathfrak{F}}_\gamma(1)$ does not satisfy the homotopy lifting property (see also [18]). In particular, we would like to gain a better understanding of periodic solutions of a linear ODE of order 3.

Finally, similar questions can be asked about curves in $\mathbb{S}^n$, $n > 2$ (γ is locally convex if $\det(\gamma(t), \ldots, \gamma^{(n)}(t)) > 0$); in [20] we show a few results about these spaces.

It would also be interesting to investigate the homotopy type of spaces of curves with bounded geodesic curvature. In [22] and [27] some first results are proved.

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