An Introduction to Geometric Topology:
Geometric Structures on Manifolds of Dimensions 2 and 3

Nicolau Corção Saldanha

Preface

This is a translated and slightly modified version of lecture notes for my mini-course on Geometric Topology in the Workshop on Topology, January 1992, PUC-Rio. The course was an introduction in six one hour lectures to a vast subject and I had to choose topics according to my personal taste. I aimed at being understood by a large audience (the course was mainly intended for graduate students), frequently mentioning results in several related areas. As an unavoidable consequence the exposition is occasionally incomplete and more than a bit superficial, many proofs were omited and others merely sketched; for many a topic to which I dedicated less than one lecture it would be hard to do justice even if I had spent the entire course discussing it. I hope to escape too harsh a judgement from the reader by giving a list of very good references; so good, in fact, that it would be foolish to try to substitute or imitate them. I expect these notes to be self-contained enough so that a graduate student can understand the concepts, the statements of theorems and the proofs (or sketches thereof) which I present. Above all, I would like to create an interest for at least some of the beautiful subjects we shall meet.

The material in part 0 is classical and good expositions can be found in many places, such as [B] and [T0]. References for part 1 are [KN] and [T0]. Parts 2 and 3 have a very rich bibliography, including the excellent books [A], [B] and [N]. Part 4 is possibly where I did least justice to the subject; for orbifolds, [T0] is a good reference as [P], [E2] and [T0] are for Fuchsian and Kleinian groups. Finally, references for part 5 are [T0], [T1] and [T2].

Nothing in these notes is due to the author. The creators or inventors of the mathematics exposed here are too many to be all listed here but I would like to mention two of them. One of them is Henri Poincaré, who in the beginning of this century gave origin to many of the ideas we talk about here and, we could well say, to the subject itself. The second one is William Thurston, which I mention for two reasons. The first is that he is a great mathematician with outstanding contributions to the subject. The second reason is more personal: his lectures at Princeton, which the author had the opportunity to attend as a graduate student, were guaranteed to create great interest.

Part 0: Hyperbolic plane and hyperbolic space

We all know the sphere and euclidean space. There is a third kind of geometry as interesting as these two but far less well known: hyperbolic geometry, which we shall now study.

Euclidean geometry has one very natural model: $\mathbf{R}^n$. Spherical geometry has an equally natural model as a subset of $\mathbf{R}^{n+1}$ with the induced metric. We will consider many interesting models of hyperbolic geometry, none of them isometrically embedded in $\mathbf{R}^N$. It is possible to find isometric copies of $\mathbf{H}^n$ inside $\mathbf{R}^N$, but only for N much greater than n; such inclusions are complicated and not particularly enlightening: we shall not study them.

One interesting model is the hyperboloid (Fig. 0.0). Consider in $\mathbf{R}^{n+1}$ the Minkowsky "metric" given by the following inner product:

$$\langle x, y\rangle = x_0y_0 + x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n.$$

Since this product is not positive definite, it does not give us a metric in the usual sense. For instance, the vector $(1, 0, \ldots, 0, 1)$ has norm 0 and the vector $(0, 0, \ldots, 0, 1)$ has norm $i = \sqrt{-1}$. Let us not worry about this right now and consider the "sphere" of radius i, i.e., the set of points with

$$x_0^2 + x_1^2 + \cdots + x_{n-1}^2 + 1 = x_n^2$$

which, as we all know, is a hyperboloid with two sheets. These two sheets are the two connected components, each corresponding to a given sign for x_n. The component with $x_n > 0$ is our first model of the hyperbolic space $\mathbf{H}^n$.

The first thing to notice about this set is that even though the inner product is not positive definite in $\mathbf{R}^{n+1}$ it is positive definite on any tangent hyperplane to the hyperboloid. This can be checked by a simple computation but not even that is necessary. Indeed, as in the case of the usual inner product, given two points x and y in $\mathbf{H}^n$ there exists a linear transformation preserving the inner product taking x to y: for instance, the reflection in the hyperplane perpendicular to the segment joining the two points (in the Minkowsky metric). This last observation together with the (immediate) observation that the inner product is positive definite on the tangent plane at $(0, 0, \ldots, 1)$ implies our first remark. In other words, the group of linear transformations which preserve the inner product and take the positive sheat to itself acts transitively on $\mathbf{H}^n$. More is true: given x and y and orthonormal basis to the tangent planes at these points, there is a unique such linear map taking x to y and one orthonormal basis to the other: this gives us an isometry of $\mathbf{H}^n$ with the same properties. As a consequence, these are the only isometries of $\mathbf{H}^n$ since two isometries with the same value and the same derivative at one point must coincide (given that $\mathbf{H}^n$ is connected). Of course, there are orientation preserving and orientation reversing isometries.

On a sphere, geodesics are given by the intersection of a plane (dimension 2) passing through the origin with the sphere. In our model of hyperbolic space geodesics are given in the same way; let us see why. It is not hard to exhibit a geodesic. Remember that

$$\sinh t = \frac{e^t - e^{-t}}{2}, \qquad \cosh t = \frac{e^t + e^{-t}}{2}.$$

Consider now the curve $\gamma(t) = (\sinh t, 0, \ldots, \cosh t)$. It is clear that $\gamma(t)$ is always on the hyperboloid. Taking derivatives we have $\gamma'(t) = (\cosh t, 0, \ldots, \sinh t)$, which has norm always equal to 1, and $\gamma''(t) = (\sinh t, 0, \ldots, \cosh t) = \gamma(t)$ which is always normal to the hyperboloid (remember it is a sphere!). Therefore, γ is a geodesic parametrized by arc length. The image of γ is the intersection of a plane with the positive sheet of the hyperboloid. From what we saw in the previous paragraph, all geodesics can be obtained from γ by applying linear transformations and our description of geodesics follows.

There is another approach, however. The isometries of $\mathbf{H}^n$ provide a large set of isometries leaving the plane fixed, and therefore the candidate geodesic also fixed; it is easy to prove that only a geodesic could admit such a large family of isometries preserving it.

A straightforward generalization of the observations in the preceding paragraph is that the intersection of any subspace with the positive sheaf of the hyperboloid is always totally geodesic. (Remember that a totally geodesic manifold is one that contains the geodesic starting at any given point with prescribed tangent vector at the point.) If non-empty, it is isometric to some $\mathbf{H}^m$. In particular, the sectional curvature is constant for any $\mathbf{H}^n$ and equal to the curvature of $\mathbf{H}^2$ which is -1.

The most widely known model of $\mathbf{H}^n$ is probably the conformal model in the unit open ball, also known as the Poincaré disk or ball (Fig. 0.1). (By the way, this model was used by M. C. Escher in, say, "Angels and devils".) In this model the space $\mathbf{H}^n$ is identified with the unit open ball with a different metric: a vector v tangent to the ball at a point x has norm

$$|v|_{\mathbf{H}} = \frac{2}{1 - |x|^2} |v|,$$

where $||_{\mathbf{H}}$ stands for the hyperbolic norm and $||$ for the usual euclidean norm. Notice that $|v|_{\mathbf{H}}$ depends on $|x|$ and $|v|$ only, not on the direction of v: this is why the model is conformal. The unit sphere (the boundary of the ball) shall be called the sphere at infinity.

In order to check that the two models are equivalent, i.e., isometric, we consider the *stereographic projection*, a construction which works for both the sphere and the hyperboloid model of hyperbolic space. For the sphere, project the points of $\mathbf{S}^n \subseteq \mathbf{R}^{n+1}$ on the hyperplane $x_n = 0$ along straight lines which go through the south pole, i.e., $(0, \ldots, -1)$. This defines a bijection between the sphere minus the south pole and the hyperplane, or, if you prefer, between the sphere and the hyperplane plus a 'point at infinity'. We claim that this projection is *conformal*, i.e., it is a similarity on any tangent plane. This can be checked by a computation, but we prefer to give a proof using elementary geometry. Let θ

be the angle between our point on the sphere and the north pole. The straight line which connects this point to the south pole makes an angle of $\theta/2$ with the vertical axis since this is an angle inscribed in a circle corresponding to an arc θ. From such reasoning we see that this line makes the same angle with the tangent plane and the horizontal plane, both angles being on a same plane but in opposite senses. The projection therefore gives a similarity between these two planes.

In the case of the hyperboloid we do the exact same thing. Notice that the south pole is on the negative sheat so the projection is defined on all of the hyperbolic plane; it is easy to see that the image is the unit open ball. Any proof that the projection gives a similarity between tangent planes carries from the sphere to the hyperboloid, not excluding our geometric proof which would when so translated involve Minkowsky geometry. It is also easy to check that the correct metric in the conformal unit ball model is indeed that which we described.

We shall now make a series of claims about these two models which can easily be verified by simple computations. As we already saw, geodesics in the hyperboloid model are given by intersections of planes passing through the origin with the hyperboloid; in the conformal unit ball model these correspond to circles or lines perpendicular to the unit sphere. The intersection of any plane with the hyperboloid gives a curve of constant curvature; these correspond to circles and lines in the unit ball model. Such curves can be classified in three kinds. First, the circle can be interior to the ball, which corresponds to a closed curve of constant curvature greater than 1. Then, the circle or line can intersect the unit sphere transversally; this corresponds to curves with constant curvature smaller than 1, geodesics being a special case. Finally, the circle can be tangent to the unit sphere which corresponds to a curve with constant curvature equal to 1: these are called *horocycles*.

More generally, we can consider intersections of arbitrary translated subspaces with the hyperboloid which correspond to round spheres of any dimension in the unit ball model; these will be submanifolds of constant curvature. The division into three cases which we saw in the preceding paragraph carries naturally to this situation. In the first case we have a sphere of any curvature. In the second case, a hyperbolic space with curvature between -1 and 0. Finally, in the case corresponding to the horocycles we have an euclidean space. In any case, intrinsic curvature equals extrinsic curvature plus the curvature of ambient space, which, in this case, is -1.

In order to get yet another model for hyperbolic space let us again consider the hyperboloid. The conformal unit ball model was obtained by projecting the hyperboloid into the plane $x_n = 0$ taking the south pole as centre. If we take the plane $x_n = 1$ and the origin instead we get the projective unit ball model (Fig. 0.2). The nice thing about this model is that geodesics become (euclidean) straight lines and isometries become projective transformations. The untiring reader can easily compute the metric for this model.

Before we see the last of classical models of $\mathbf{H}^n$, let us consider some transformations of euclidean space. An *inversion* of (extended) euclidean space with respect to a sphere of radius R is the transformation which takes a point at a distance r from the center of the sphere to the point lying on the same ray from the centre but which lies at a distance

R^2/r from the centre of the sphere. We shall also decree that the inversion takes the centre of the sphere to the "point at infinity", which is itself taken to the centre of the sphere. Classical geometry teaches us that inversions take circles and lines to circles and lines and that it is a conformal transformation, i.e., it respects angles. We define the group of Möbius transformations as being generated by similarities and inversions, so that it is a group of conformal bijective maps from euclidean space plus the point at infinity to itself. Since inversions change orientations, there are orientation preserving and orientation reversing Möbius transformations.

Let us now apply an inversion to the conformal unit ball model; notice that this must give us a conformal model. If the centre of inversion lies outside the ball we have only a minor modification of the original model: a conformal model in another ball. If the centre of inversion lies intside the ball we have a model of hyperbolic space minus one point in the exterior of a sphere. If, however, the centre of inversion lies on the boundary of the ball we get something more interesting (Fig. 0.3). The interior of the ball shall be taken to a half-space, say the upper half-space. Geodesics will be circles with centre on the boundary or vertical straight lines. Moreover, the metric shall be very simple; let v be a vector tangent to the upper half-space at a distance r from the boundary: we have

$$|v|_{\mathbf{H}} = \frac{1}{r}|v|.$$

In this model, as well as in the conformal unit ball model, isometries of hyperbolic space correspond precisely to Möbius transformations leaving the associated region invariant.

Let us now leave aside for a moment hyperbolic space in arbitrary dimension and take a closer look at $\mathbf{H}^2$, the hyperbolic plane. The unit ball or upper half space can now be taken as subsets of the complex plane $\mathbf{C}$.

Define a *Möbius function* in $\mathbf{C}$ as

$$f(z) = \frac{az+b}{cz+d}$$

where a, b, c and d are arbitrary constants with $ad - bc \neq 0$. This last condition guarantees that f is well defined (except for a pole) and non-constant. Notice furthermore that f and g with coefficients a, b, c, d and a', b', c', d' respectively are equal if and only if there is $\lambda \neq 0$ such that $a' = \lambda a$, $b' = \lambda b$, $c' = \lambda c$, $d' = \lambda d$. We may assume without loss of generality that $ad - bc = 1$. If we represent a Möbius function by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, composition or inversion of functions correspond to multiplication or inversion of matrices. The group of Möbius functions is therefore $PGL(2, \mathbf{C}) = PSL(2, \mathbf{C})$, the group of complex 2×2 matrices (of determinant 1), where matrices equal up to a scalar multiplicative factor are identified.

The coincidence of names is no accident: Möbius functions in $\mathbf{C}$ are precisely the same as orientation preserving Möbius transformations in $\mathbf{R}^2$. The consequences are clear: orientation preserving isometries of $\mathbf{H}^2$ correspond to Möbius functions leaving the unit disk or the upper half plane invariant, according to the model. This second approach gives an algebraic description of the group of orientation preserving isometries of $\mathbf{H}^2$: this is $PSL(2,\mathbf{R})$, since it is easy to see that a Möbius function preserves the upper half plane precisely when its coefficients are real (if we demand that the determinant be 1). Actually, we can also see that the full group of isometries of $\mathbf{H}^2$ corresponds to $PGL(2,\mathbf{R})$ (hint: consider $z \mapsto \overline{z}$).

Suppose f is a holomorphic bijection from the unit disk to itself. We shall see in part 2 that f must be a Möbius transformation.

Consider $\mathbf{H}^n$ as the upper half space. An isometry of $\mathbf{H}^n$ is a Möbius transformation preserving this half space and therefore inducing a Möbius transformation on the hyperplane of dimension $n-1$; it is easy to see that this defines an isomorphism between the isometries of $\mathbf{H}^n$ and the Möbius transformations of $\mathbf{R}^{n-1}$. In particular, for $n=3$, this gives a bijection between orientation preserving isometries of $\mathbf{H}^3$ and the group of Möbius functions in $\mathbf{C}$, which, as we already saw, is $PSL(2,\mathbf{C})$.

Let us return to the hyperbolic plane to see yet another model. Consider in $\mathbf{R}^2$ the space of positive definite inner products which induce the usual area form. This is the space of 2×2 real symmetric matrices $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $ac - b^2 = 1$ e $a+c>0$. If we write $a = x_2 + x_0$, $b = x_1$ e $c = x_2 - x_0$ the above conditions become $x_0^2 + x_1^2 + 1 = x_2^2$ e $x_2 > 0$, the hyperboloid about which we have already talked so much. Area preserving linear transformations act naturally on this set of inner products thus giving us a different interpretation for the identification of the group of orientation preserving isometries of $\mathbf{H}^2$ with $PSL(2,\mathbf{R})$. Therefore, given a manifold of dimension 2 with a riemannian metric, metrics with the same area form correspond to sections of a bundle where each fibre is a hyperbolic plane representing the various inner products with the same area form.

After all these models, let us consider the hyperbolic plane quite independently from any model. In the euclidean plane, the sum of the internal angles of any triangle is always equal to π (or 180°). In the sphere, the sum of the internal angles is always greater than π; the Gauss-Bonnet theorem actually tells us that this sum is $\pi + S$, where S is the area of the triangle. In the hyperbolic plane, the sum of the internal angles is always less than π; again, by the Gauss-Bonnet theorem, this sum is $\pi - S$, S being the area of the triangle.

These simple observations give us some interesting consequences. We see, for instance, that the area of a (bounded) triangle is always strictly less than π. A degenerate triangle with three vertices on the circle at infinity has area equal to π. All such degenerate triangles are equal: there always exists a unique isometry taking one to the other if we prescribe which vertex should go to which vertex. More generally, triangles with equal angles are congruent, quite unlike euclidean geometry, and given three angles with sum smaller than π it is always possible to construct a unique triangle with these angles.

We have many ways of tiling hyperbolic plane; many more than with euclidean plane. Take, for instance, triangles with angles $\pi/2$,$\pi/3$ and $\pi/7$, which gives us tiles of area $\pi/42$; it is easy to tile hyperbolic plane with such triangles. There exist many other possibilities, such as regular hexagons with right angles, regular octagons with all angles equal to $\pi/4$ (Fig 0.4). The reader may have noticed that in all examples the minimum area for the tile was $\pi/42$, as in the first example. This is indeed true for any tiling, provided, of course, all tiles are congruent; the proof of this fact is not hard but we shall not give it here.

We get an interesting tiling by using degenerate triangles. Since sides are infinite, there is more than one way they can be glued. Notice first of all that different ways of glueing *are* different and that there is one canonical way of doing it since on each side we have a privileged point: the basis of the height. Let us first consider the tiling obtained by making such privileged points coincide when glueing two tiles. This picture is specially interesting in the upper half plane model (Fig 0.5). Let the first triangle have vertices 0, 1 and ∞; other tiles will have rational vertices. The way this works is closely related to Farey fractions: two rationals $\frac{a}{b} < \frac{c}{d}$ in irreducible form are joined by an edge if and only if $cb - da = 1$; we make $\infty = \frac{1}{0}$. We therefore have a triangle with vertices $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$, all fractions irreducible, if and only if $cb - da = eb - fa = ed - fc = 1$; this implies $c = a + e$ e $d = b + f$.

What happens if we glue tiles in some other way? If the tiling is to be regular we must have the new basis of the height at a fixed distance x to the right (left) of the old one (notice the terminology 'right' and 'left' works in such a way that it does not matter which triangle is old and which is new). The surprising thing is: such a tiling does *not* cover the whole hyperbolic plane. A picture or a few computations in any model (we sugest the upper half plane) should be enough to convince the reader.

We would like to conclude this part with a simple remark opening the gates to a huge territory about which we will talk a little later on. A tiling of hyperbolic plane gives us a discrete group of isometries which is the same as a discrete subgroup of $PSL(2, \mathbf{R})$. Likewise, a tiling of hyperbolic space gives us a discrete group of isometries of $\mathbf{H}^3$ or a discrete subgroup of $PSL(2, \mathbf{C})$. These are called Fuchsian and Kleinian groups.

Part 1: Geometric structures on manifolds

We know three kinds of geometry: spherical, euclidean and hyperbolic. It is easy to see that these are, up to scale, the only three riemannian geometries of constant sectional curvature; in particular, the only ones for which there exists an isometry taking any point with any orthonormal basis on the tangent plane to any other point with any other such basis. Later we shall meet other kinds of geometries but in this section we want to see how a manifold can have geometric structures of several sorts. All manifolds are assumed to be connected except if the opposite is said.

Let us begin by recalling the usual definition of a manifold. A manifold N is a topological space equipped with a family of *charts*. A chart is a homeomorphism between an open subset of a model space M (usually $\mathbf{R}^n$) and an open subset of N. We demand that N should be covered by these open sets. At this point, it makes no sense to ask if a chart is, say, C^∞. If we want to define a C^∞ manifold it is thus important to check not if individual charts are nice but if charts are compatible.

More precisely, if $f_i : A_i \to V_i$ and $f_j : A_j \to V_j$ are charts with $A_i, A_j \subseteq M$ and $V_i, V_j \subseteq N$ we can compare charts in $V_i \cap V_j$, that is, we can take $\gamma_{ij} : f_i^{-1}(V_i \cap V_j) \to f_j^{-1}(V_i \cap V_j)$ defined by $\gamma_{ij} = f_j^{-1} \circ f_i$. The function γ_{ij} goes from a subset of M to M and since we assume M to be a well known model space it does make sense to ask whether it is "nice". In the classical situation, we say that N is a C^∞ manifold if each γ_{ij} is itself C^∞.

The notion of pseudogroup generalizes these ideas. A pseudogroup Γ is a set of functions from open sets in M to open sets in M. Different pseudogroups will give us different kinds of structures on manifolds. Let us see the formal definition of pseudogroup which shall be used in many examples. By the way, pseudogroups on a topological space M generalize the notion of a group of homeomorphisms of M, hence the name.

A pseudogroup Γ on a topological space M is a set of homeomorphisms between open subsets of M such that the following properties hold:

- If $\mathrm{Id} : M \to M$ is the identity, then $\mathrm{Id} \in \Gamma$.
- If $f \in \Gamma$ then any restriction of f to an open set is also in Γ.
- If $U = \bigcup_{i \in I} U_i$, $f : U \to V$ is a homeomorphism and if for all $i \in I$ we have $f|_{U_i} \in \Gamma$ then $f \in \Gamma$.
- If $f : U \to V$ and $g : U' \to V'$ belong to Γ and $V \cap U' \neq \emptyset$ then $g \circ f$ defined from $f^{-1}(V \cap U')$ to $g(V \cap U')$ belongs to Γ.
- If $f : U \to V$ belongs to Γ then $f^{-1} : V \to U$ also belongs to Γ.

We shall now see several examples of the following kind: we define a pseudogroup and consider manifolds with charts such that $\gamma_{ij} \in \Gamma$ for all i, j. We shall than try to understand the geometric implications of having such a system of charts.

A simple example is to take the pseudogroup of all C^∞ diffeomorphisms from an open subset of $\mathbf{R}^n$ to another. This gives us precisely the notion of a C^∞ manifold. We can now tell when a function from a C^∞ manifold to another is C^∞: simply compose it with the charts. Likewise, we can take the pseudogroup of all C^r diffeomorphisms, which will give us the notion of a C^r manifold. If we take as model space $\mathbf{C}^n$ and consider the pseudogroup of all complex analytic diffeomorphisms from one open set to another we get the notion of complex manifold.

We have a completely different example if we take the pseudogroup of translations in $\mathbf{R}^n$, i.e., the set of homeomorphisms from one open set to another which are translations when restricted to any connected component of the domain. This extra complication is forced on us by the definition of pseudogroup; from now on we shall not be so careful with this issue. A manifold for this pseudogroup comes with a local action of $\mathbf{R}^n$; the action is locally free so that if it is complete (in particular, if the manifold is compact) the manifold is the quotient of $\mathbf{R}^n$ by a discrete subgroup. More generally, if we take as model space a Lie group we can consider the pseudogroup of homeomorphisms which, when restricted to a connected component, are left (right) multiplications by some fixed element. As in the previous example, compact manifolds with this kind of structure are quotients of the original group by discrete subgroups. It is necessary in order to guarantee that we have a quotient that the manifold be complete (in a sense to be explained); otherwise, we can produce complicated examples.

The most interesting examples for us, however, are neither as loose as in the case of C^r manifolds nor so rigid as this last example. Taking $\mathbf{R}^{n+m}$ as the model space, let Γ be the pseudogroup consisting of C^∞ functions where any of the first n coordinates of $f(x)$ depend on the n first coordinates of x only. If we call images of $\{*\} \times \mathbf{R}^m$ by the charts *leaves* we see that leaves are well defined, independently of charts: a leave is a set of points where the first n coordinates are held fixed but the m remaining can vary. What we have here is a manifold with a foliation, leaves having dimension n.

Let us now consider a model space with a metric, such as euclidean, spherical or hyperbolic space. If we take the pseudogroup of isometries, we have the concepts of euclidean, spherical and hyperbolic manifols. This is equivalent to giving the manifold a riemannian metric with constant sectional curvature equal to 0, 1 or -1, respectively. These are going to be some of our main examples.

Let us take a look at what happens in dimension 2. We know that the compact orientable surfaces are the sphere, the torus, the two-holed torus or sphere with two handles, and, in general, spheres with any number of handles. Likewise, the non-orientable surfaces are the projective plane, the Klein bottle, and, in general, connected sums of projective planes. For any of these surfaces, suppose it has one of these three geometric structures. The Gauss-Bonnet theorem tells us that the integral of the curvature over the whole surface equals 2π times the Euler characteristic of the surface; this does not depend on orientability. As a consequence, if the Euler characteristic is positive there can clearly be no euclidean or hyperbolic struture: all we can hope for is a spherical structure (although it is not clear that such a structure should exist); likewise, if the Euler characteristic is zero or negative we can only hope for an euclidean or hyperbolic structure, respectively. Obviously, the sphere and the projective plane admit spherical structures and the torus and Klein bottle admit euclidean structures. All other compact surfaces have negative Euler characteristic: we would like to know whether they admit hyperbolic structures.

In order to settle this question, consider the traditional way of constructing these manifolds. Take a polygon with $d = 4 - 2\chi$ sides, χ being the Euler characteristic of the intended manifold, and glue pairs of edges in a certain way; all vertices are glued to one point. If we use for the construction a regular polygon in hyperbolic plane all of its angles being equal to $2\pi/d$ (the existence of such a polygon is clear) the geometry will be respected by the glueing procedures so that the result will be a hyperbolic structure on the surface.

Other interesting pseudogroups are given by similarities, affine, projective and Möbius transformations in $\mathbf{R}^n$. Notice that there are many inclusions among our examples of pseudogroups. Thus, for instance, an euclidean manifold automatically gets a similarity structure since any euclidean isometry is a similarity. Likewise, a similarity structure implies both an affine and a Möbius structure; an affine structure implies a projective structure. A spherical structure implies both a Möbius and a projective structure via stereographic and central projection, respectively; for the same reason, a hyperbolic structure also implies both a Möbius and a projective structure.

Let us now introduce a construction common to many of these structures. As an example, let N be a connected hyperbolic manifold and let $\tilde{N}$ be its universal covering. Take x_0 to be some point of $\tilde{N}$; by hypothesis, there is some chart defining an isometry from a connected neighborhood of x_0 to some open set in $\mathbf{H}^n$; we want to extend this map so that it is defined on all of $\tilde{N}$. If it already is, we are done. If not, consider some other similar chart whose domain intersects the domain of the old one; we want to compose this new chart with an isometry of $\mathbf{H}^n$ so that it becomes compatible with the previous one in order to extend the domain of our map. If this can be done, the process can be repeated until the map is defined on all of $\tilde{N}$. Our only worry in this process is that it must be possible to make things compatible at each step; this will happen if the intersection of the domain of the new chart with the current domain of the map (which is the union of domains of all previous charts) is always connected. It is a known fact (from algebraic topology) that it is possible to choose the domains of charts satisfying the property above; this happens in $\tilde{N}$ since it is simply connected but usually not in N. This defines a locally bijective map from $\tilde{N}$ to $\mathbf{H}^n$ which is unique up to composition with isometries of $\mathbf{H}^n$; this will be called the *developing map*.

Continuing our construction, $\pi_1(N, x_0)$ acts naturally on $\tilde{N}$. Indeed, recall the construction of $\tilde{N}$ as the space of all classes of continuous paths in N starting at x_0 where paths are identified under homotopy with fixed endpoints; the projection from $\tilde{N}$ to N is the endpoint of any path in the equivalence class. With this interpretation, the action works by taking $p \in \pi_1(N, x_0)$ and $x \in \tilde{N}$ to the path obtained by first following some path in the equivalence class of p and then some path in the equivalence class of x. We can compose this action with the developing map in order to obtain an action of $\pi_1(N, x_0)$ on $\mathbf{H}^n$ by isometries, or, equivalently, a homomorphism from $\pi_1(N, x_0)$ to isometries of $\mathbf{H}^n$. This action, which we shall call the *holonomy*, tells us what effect we feel from the point of view of hyperbolic structure as we go around some loop p.

There was nothing special about hyperbolic structure in this construction: nothing would have changed if we had a spherical, euclidean, similarity, affine, projective or Möbius structure but the construction definitely does not work for, say, C^∞ manifolds. Let us see what has to be assumed about structures in order to define developing maps and holonomy. Each element of the pseudogroup with connected domain should be the restriction of a unique homeomorphism from model space to itself which also lies in the pseudogroup: this will guarantee the existence and uniqueness of the new map in the process of extending the domain in order to build the developing map. In other words, the elements of the pseudogroup should be restrictions of a group of homeomorphisms of model space with the property that distinct elements of the group never coincide on an open set. In particular, this will hold if all elements of the group are analytic; in this case, the developing map is nothing more than analytic continuation.

We saw that the developing map is a locally bijective function from $\tilde{N}$ to the model space but it need not be bijective (we assume model space simply connected). Trivial examples of this are bounded open subsets of $\mathbf{R}^n$ with the induced euclidean structure. A more interesting example is obtained by taking any quadrilateral in $\mathbf{R}^2$ and glueing opposite sides to get a torus with similarity structure; the developing map will be bijective if and only if the quadrilateral is a parallelogram. A manifold with a structure which allows the construction of a developing map will be called *complete* if the developing map is bijective. This definition coincides with the one from riemannian geometry in the case of euclidean or hyperbolic structures. A complete manifold is a quotient of model space by some discrete group. We shall be mainly interested in complete manifolds.

We have by now many definitions and we can begin to ask questions, some of them hard, many of them open. As an example of a non-trivial question we can answer, let us find out which compact surfaces admit affine structures. The torus and the Klein bottle clearly do; the sphere and the projective plane do not: such a structure would produce as developing map a locally bijective function from the sphere to the plane, which is absurd. The surfaces of negative Euler characteristic also do not admit affine structures, and we shall now present a sketch of a proof of this fact. In higher dimensions, it is an open problem whether the existence of affine structure implies zero Euler characteristic.

Suppose by contradiction that N is a two-holed torus with an affine structure. The argument for other orientable surfaces is similar; if a non-orientable surface admits an affine structure so does its double cover which is orientable. Let x_0 be any point of N; join x_0 to itself by four curves which go around N somehow in such a way that they do not intersect and that N minus these curves is an octagon, the curves being the sides. We can therefore think of N as coming from an octagon in the plane where sides were glued in the usual way. The glueing process always corresponds to an affine transformation.

In the plane containing the octagon, take two constant linearly independent vector fields. We shall now attempt to give a basis to the tangent plane at each point of N. If we exclude x_0 and the four curves, this can easily be done by lifting the above vector fields to N. It is not hard to change these fields only near the four curves and so define the vector fields everywhere except on x_0. In order to do this, remember glueing is done by affine transformations. Therefore, upon crossing one of the four curves in a certain sense the two vector fields suffer the action of a fixed orientation preserving linear transformation. If we take some fixed path from the identity to this linear transformation, this gives us a prescription on how to smooth out the transition in the neighborhood of the curve. This prescription, which is constant on each curve, is given by an element of G, the universal cover of $GL^+(2,\mathbf{R})$.

Let us now take a look at a neighborhood of x_0: how do our vector fields behave if we follow a small closed curve around x_0? We know what pattern of curves to expect: we cross curve a in the positive sense, then b in the positive sense, a in the negative sense, b in the negative sense, then similarly for c and d. Our vector fields shall therefore be multiplied by $D^{-1}C^{-1}DCB^{-1}A^{-1}BA$, a product of two commutators in G. On the other hand, since the Euler characteristic of N is -2 and x_0 is the only singularity, the vector fields should turn around twice. The element of G corresponding to two full turns would therefore be a product of two commutators. But this is not the case, as we now show: even though this element is a product of a larger number of commutators, two are not enough.

Let A and B be arbitrary elements of G; we shall show that the commutator of A and B, $B^{-1}A^{-1}BA$, turns no vector by 2π or more; this clearly finishes the proof. The elements of G corresponding to an integer number of half turns are in the centre. Multiplying A and B by such elements leaves the commutator of A and B unchanged. We can therefore assume that both A and B turn any vector by less than π. Suppose either A or B has a real eigenvalue, which must be positive. We can show a vector which turns by less than π, which clearly implies that all vectors turn less than 2π: this vector can be an eigenvector of A or the inverse image by A of an eigenvector of B. We have only the case of no real eigenvalues left: in this case A and B turn all vectors in the same direction so that two of the four terms of the commutator turn vectors one way, the other two turn them the opposite way. Since each term turn any vector by less than π, all vectors will be turned by less than 2π.

Before we leave the subject of affine manifolds, we make another observation: even though N, the two holed torus, admits no affine structure, $N \times S^1$ does. Indeed, we already know that N admits a hyperbolic structure. Think of this in the hyperboloid model inside $\mathbf{R}^3$: isometries are linear maps and if we take an extra radial dimension it is easy to see how to give affine structures to $N \times \mathbf{R}$ or $N \times S^1$. Notice that these affine structures are not complete.

Nice as they are, pseudogroups are not always the best or the only way to give structures to manifolds. Another more classical way to do so is to give some tensor field on the manifold, such as a metric, a symplectic form or a volume form. A simple extension of this idea gives us our next, very important example: conformal manifolds.

Part 2: Complex structures

We have seen several kinds of structure which a manifold admits. We now change our point of view, going from the general to the particular, paying more attention to a few special kinds of structures and restricting ourselves to dimension 2. We shall see how spherical, euclidean or hyperbolic structures relate to complex and conformal structures.

Recall a few facts from complex analysis. The maximum principle says that if f is holomorphic on a connected open region D and continuous in $\bar{D}$ then the absolute value of f assumes its largest value on the boundary of D. Furthermore, if this maximum value is also reached at some interior point than f is constant. Proofs of this fact are exposed in several books and we do not repeat them here.

A classical corollary of the maximum principle is Schwarz lemma. Let $\Delta \subseteq \mathbf{C}$ be the unit open ball and let $f : \Delta \to \Delta$ be a holomorphic function with $f(0) = 0$. Then $|f'(0)| \leq 1$ and, for all $z \in \Delta$, $|f(z)| \leq |z|$; furthermore, if equality ever holds we must have $f(z) = az$ where a is a complex number of absolute value 1. The proof is easy: define

$g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$, notice that g is holomorphic and apply the maximum principle to g.

As we saw in part 0, Δ is a model of hyperbolic plane (we mean the conformal model, not the projective one). We can therefore talk about hyperbolic distance between points of Δ, hyperbolic length of arcs or hyperbolic area of subsets of Δ. Remember that the orientation preserving isometries of Δ are the (holomorphic) Möbius functions that take Δ to itself. Schwarz lemma admits a nice formulation (due to Pick) in terms of hyperbolic geometry. For any holomorphic function f from Δ to itself and for any points z and w in Δ, the hyperbolic distance between $f(z)$ and $f(w)$ is less than or equal to the hyperbolic distance between z and w. In order to prove this, compose f with Möbius functions so that it takes 0 to 0 and use the known formulation. Similarly, any such f must reduce or leave unchanged hyperbolic lengths of arcs and hyperbolic areas (integrate) and if any of these is left unchanged f must be a Möbius function (a hyperbolic isometry).

We are now ready to settle a question which we left unanswered in part 0: A holomorphic bijection from Δ to Δ has to be a Möbius function. Indeed, both f and f^{-1} reduce hyperbolic distances, which means that f is an isometry; we already know that isometries are Möbius functions.

Given two points z_0 and w_0 in Δ there always exist holomorphic functions from Δ to itself taking z_0 to w_0 Given two pairs of points z_0, z_1 and w_0, w_1, Schwarz's lemma tells us that it is impossible to take z_i to w_i if the hyperbolic distance between w_0 and w_1 is greater than the hyperbolic distance between z_0 and z_1. It is easy to see that this condition is actually necessary and sufficient. The Schwarz-Pick theorem generalizes this to any number of points. Let z_k be n points in Δ and let w_k be another such n points. A holomorphic function f from Δ to Δ taking z_k to w_k exists if and only if the Hermitian form in $\mathbf{C}^n$ given by

$$Q(t) = \sum_{\substack{0 \le k < n \\ 0 \le h < n}} \frac{1 - w_h \bar{w}_k}{1 - z_h \bar{z}_k} t_h \bar{t}_k$$

is positive definite. We shall not prove this theorem. Notice, however, that it is not hard to get a recursive algebraic necessary and sufficient condition: without loss of generality, by applying Möbius functions, $z_0 = w_0 = 0$. The putative f can now be divided by z and remain holomorphic from Δ to Δ (unless it becomes constant with absolute value 1); it should now take z_k to w_k/z_k ($k \neq 0$). We therefore reduced the problem with n numbers to the problem with $n-1$ numbers. It remains to show that the condition thus obtained is equivalent to that mentioned in the statement of the theorem.

Let us now recall another classical theorem in complex analysis: the Riemann mapping theorem. Let $D \subseteq \mathbf{C}$, $D \neq \mathbf{C}$ be an open, connected and simply connected set. The theorem states that there is a holomorphic bijection between D and Δ. Notice that since we know all holomorphic bijections from Δ to itself, once we know a holomorphic bijection between D and Δ, we know all of them. The proof of the theorem becomes clearer if we take $x_0 \in D$ and say it must be taken to 0 with real positive derivative.

Let us first see that there is some injective holomorphic function from D to Δ taking x_0 to 0. If D is bounded, this is easy: take a function of the form $f(z) = az + b$. If the complement of D contains an open set a Möbius function will take D to a bounded set, reducing the problem to the previous case. Finally, consider the general case: there is some complex number not in D, say, without loss of generality, 0. Since D is simply connected we can define $\log(z)$ on D; this defines a bijection between D and some set the complement of which contains an open set, again reducing the problem to the previous case.

Let us now consider the class of all holomorphic injective functions f from D to Δ taking x_0 to 0 with $f'(x_0) \in \mathbf{R}$, $f'(x_0) > 0$. We want to take a function in this class with maximum $f'(x_0)$. It has to be shown that this is possible, that is, that the maximum value is reached: this can be done by using the theorem of Arzelá-Ascoli with the supremum norm over some compact subset of D, our class of functions being bounded and equicontinuous. Let therefore f be some function in the above class with maximum derivative: we claim that f is bijective. Indeed, assuming w in Δ but not in the image of f, we shall construct g in our class of functions with $g'(x_0) > f'(x_0)$. Compose f with some Möbius function preserving Δ, taking w to 0 and 0 to some positive real number; next compose the result with $q(z) = \sqrt{z}$ (this is possible because now 0 is not in the image); finally, define g by composing the result with another Möbius function in such a way that $g(x_0) = 0$, $g'(x_0) > 0$. The fact that $g'(x_0) > f'(x_0)$ follows from a simple computation. This finishes the proof; notice that the proof provides us with a construction of good approximations of the desired bijection f.

A manifold with a *conformal structure* is a manifold with a riemannian metric, but two such metrics are identified when one is obtained from the other by multiplication by a scalar function. This means that we know how to measure angles between vectors in the same tangent plane but not lengths of vectors.

A related but different concept is that of a flat conformal structure. This is defined by the pseudogroup of conformal diffeomorphisms of open subsets of $\mathbf{R}^n$ (the model space); a diffeomorphism is said to be conformal when its derivative at each point is a similarity. Möbius structures induce flat conformal structures (and therefore hyperbolic, euclidean or spherical structures do as well). Somewhat surprisingly, in dimension greater than 2, flat conformal structures are actually equivalent to Möbius structures: this comes from the fact that the only conformal diffeomorphisms of (even small) open subsets of $\mathbf{R}^n$, $n > 2$, are Möbius transformations.

A flat conformal structure clearly induces a conformal structure. It is not clear, however, when a conformal structure comes from a flat conformal one; this happens only if there are conformal diffeomorphisms between small open sets in the manifold and small open sets in $\mathbf{R}^n$. This, by the way, explains the word 'flat': a conformal structure is flat if it is locally just like $\mathbf{R}^n$. The problem of finding out whether a conformal structure is flat is therefore similar to the problem of finding out whether a riemannian metric is flat. This last problem is solved by the curvature tensor R: the metric is flat iff this tensor (which is defined in terms of the metric) is zero. It turns out that something similar can be done for conformal structures. If we consider any metric compatible with the conformal structure and compute its curvature tensor R it is possible to extract from R a 'conformal curvature' tensor which depends only on the original conformal structure and not the choice of the metric. This tensor plays a role very similar to R, but we shall not describe its construction here, much less prove any of its properties.

By the way, something similar happens for complex structures: what we have defined using pseudogroups is analogous to flat structures in the above discussion. We could define pseudocomplex structures as given by a smooth family of linear transformations J from each tangent plane to itself satisfying $J^2 = -\mathrm{Id}$ (this corresponds to multiplication by i). Asking when a pseudocomplex structure comes from a complex structure is somewhat similar to asking when a metric or a conformal structure are flat. Indeed, here again there exists an analogue of the curvature tensor but we shall discuss none of this for high dimensions. In dimension 2 (complex dimension 1), as we shall see, a pseudocomplex stucture always comes from a complex structure.

Again in dimension 2, a pseudocomplex structure is equivalent to a conformal structure plus orientation: given a conformal structure and orientation we define J as a counter-clockwise rotation of $\pi/2$ and conversely, given J, the conformal structure and orientation are easily constructed.

Complex manifolds of real dimension 2 (complex dimension 1) are called Riemann surfaces. The Riemann mapping theorem says that any open, connected, simply connected proper subset of $\mathbf{C}$ is equivalent as a complex manifold to Δ, the unit disk. On the other hand, the plane is not equivalent to Δ since a bounded holomorphic function defined on the whole plane must be constant. This gives a complete classification of connected, simply connected complex manifolds of dimension 1 which can be included in $\mathbf{C}$; it would of course be interesting to have a comparable result without this last very limiting condition. There is at least one other example of a connected, simply connected Riemann surface: the so called Riemann sphere, obtained from $\mathbf{C}$ by adjoining a point at infinity. To be more precise, we can give a complex structure to the sphere by using two charts, each one covering the sphere minus a pole, each one given by stereographic projection from a pole. The Riemann sphere is obviously not equivalent to either the plane or the disk: it is not even homeomorphic. The *uniformization theorem* says that these are the only examples: any connected, simply connected Riemann surface is equivalent to the disk, the plane or the Riemann sphere. This is a hard theorem and we shall not attempt to give even a sketch of a proof; we shall however use it quite a bit.

An immediate consequence of the uniformization theorem is that the universal covering of any Riemann surface is either the sphere, the plane or the disk. These are the classical models for spherical, euclidean and hyperbolic geometry. It is trivial to get a complex structure from a hyperbolic, euclidean or spherical structure; we now see that to a large extent the opposite also holds. If the universal covering of a Riemann surface is a sphere, the surface is the sphere itself and admits a complete spherical structure which is unique up to conjugation by a Möbius function; more naturally, it has a unique complete Möbius structure. If its universal covering is a plane, the surface is either the plane itself, a cylinder or a torus; in any case, it gets a complete euclidean structure which is unique up to change of scale; more naturally, it has a unique complete similarity structure. For any other surface, the universal covering is a disk; the surface thus admits a unique complete hyperbolic structure, the metric corresponding to this structure being called the Kobayashi metric.

There is one step missing: we still have to relate pseudocomplex or conformal structures in real dimension 2 to complex structures. The uniformization theorem generalized the Riemann mapping theorem by classifying all connected, simply connected Riemann surfaces. We now want a further generalization which applies to surfaces with conformal structures. Such a generalization exists, and in it we even drop the condition that the conformal structure must change smoothly and substitute by what is almost the weakest possible assumption, namely, that the conformal structure is measurable. The so called measurable Riemann mapping theorem now states that for any connected, simply connected surface of dimension 2 with a conformal structure which changes measurably there exists a bijection between the surface and the sphere, plane or disk (with the usual complex structure) which respects the conformal structure almost everywhere (in the measure theoretic sense). This again is a hard and very important theorem which we will use but whose proof we shall not discuss. Its corollary is that in dimension 2 giving a complex structure is the same as giving a conformal structure. This finishes the picture and we now know how to relate several kinds of structures on manifolds of dimension 2.

Part 3: Teichmüller space

We saw how closely related spherical, euclidean, hyperbolic, complex and conformal structures are for manifolds of dimension 2. We shall now consider the possibility of giving different complex structures to a fixed manifold. Teichmüller space is, in a sense, the space of "really" distinct complex structures a manifold may have. Until further notice, all manifolds are going to be of dimension 2.

Let us consider a simple problem. Given two open annuli in the complex plane, when are they equivalent, i.e., when does there exist a holomorphic bijection between the two? As we saw, each one has its Kobayashi metric corresponding to its unique complete hyperbolic structure. (Here we exclude the cylinder, or plane minus one point, for being euclidean, not hyperbolic.) Consider simple closed curves inside each annulus which are not boundaries of disks contained in the annulus, that is, curves which go once around the annulus. There are two cases to account for depending on the geometry of the annulus. If it is a simply connected region minus one single point this set of curves contains arbitrarily short ones, but no closed geodesic. Otherwise, among these curves there is a unique geodesic.

These claims can be proven in many ways: we present three arguments which we deem interesting. One idea is to look at a simple closed curve as above and at its lifting through the developing map to the hyperbolic plane. It turns out that this lifted curve tends to one or two points in the circle at infinity, according to our classification of annuli given above. This follows (in a correct and intuitive but non-obvious way) from the crucial observation that points which are far away along the curve correspond to points which are far away in hyperbolic plane. The same must then hold for any other curve: the limit points do not depend on the choice of the curve since the distance between two such lifted curves is uniformly bounded. In case there are two points, there is a unique hyperbolic straight line connecting them and this is the lift of our closed geodesic. Otherwise, horocycles tangent to the unit circle at our unique limit point will project to arbitrarily short curves (Fig. 3.0).

A second proof comes from considering the holonomy of the annulus. Since its fundamental group is $\mathbf{Z}$, a look at isometries of $\mathbf{H}^2$ will show that we must have, in the first case, a group of (parabolic) isometries 'around' a point on the circle at infinity; otherwise, we have a group of translations along a certain axis. In the second case, the axis is invariant by the group and will correspond to the desired closed geodesic. This kind of analysis of groups of isometries of $\mathbf{H}^2$ shall be considered later.

The third idea is probably the most intuitive: look for the shortest curve in this class. This can be done by a compactness argument once we show that curves that enter the regions near the boundaries of the annulus are always long: this happens in the second case only. This curve must be a geodesic. Uniqueness is easy: two geodesics would give us either a hyperbolic bigon (if they intersect) or a hyperbolic annulus with boundary (if they don't); neither of these entities can exist according to the Gauss-Bonnet theorem. (A *bigon* is a two-sided polygon.)

Whichever argument we prefer, the fact is that the length of this unique closed geodesic is invariant by holomorphic bijections. The two annuli can only be equivalent if the two lengths are equal. In this situation, they are indeed equivalent and we can construct an isometry starting from the closed geodesic. Start by mapping one geodesic to the other. The annulus is the disjoint union of all geodesics perpendicular to the unique closed one: extend the map (by isometry) along these. The hyperbolic structure guarantees that the map so constructed is an isometry.

There are physical interpretations of this invariant. Imagine the annulus made of conducting material: we can connect one wire to each boundary component and measure the resistence R; the length of the geodesic will be π/R. If we prefer, we can make the annulus from isolating material and build with it a capacitor again connecting a wire to each boundary component; the capacity is obtained dividing the length of the geodesic by π.

Similar ideas to those presented in the last paragraphs would give us invariants for other hyperbolic Riemann surfaces. Before doing this let us define, for compact manifolds the *Teichmüller space* which classifies such structures; we restrict ourselves to compact surfaces for simplicity. Let M be a compact manifold of dimension 2. Consider the class of all smooth conformal structures on M. Any diffeomorphism $f : M \to M$ induces a map from this class of structures to itself, always taking a structure to another equivalent to it via f; a smooth isotopy from f to the identity defines a smooth family of equivalent structures. If in this class of structures we identify two conformal structures which are equivalent to one another by a diffeomorphism isotopic to the identity, we get the Teichmüller space of M.

Let us compute the Teichmüller space for a few simple examples. For the sphere, it is a consequence of the measurable Riemann mapping theorem that any two structures are equivalent. It is well known, on the other hand, that all orientation preserving diffeomorphisms of the sphere are isotopic to the identity. It follows that the Teichmüller space of the sphere is a point.

The next example is the torus. Let us look at $H_1(\mathbf{T}^2, \mathbf{R}) = \mathbf{R}^2$: this vector space comes with a natural area form. Remember that a diffeomorphism from the torus to itself is isotopic to the identity if and only if it induces the identity map in $H_1(\mathbf{T}^2)$. We saw that a conformal structure on the torus defines a unique euclidean structure of total area 1. This euclidean structure gives us on $H_1(\mathbf{T}^2, \mathbf{R})$ an inner product compatible with the already defined area form; clearly, if two structures are equivalent by isotopy they induce the same inner product. Conversely, such an inner product on $H_1(\mathbf{T}^2, \mathbf{R})$ defines an euclidean structure on $\mathbf{T}^2$ up to equivalence by isotopies. We saw in part 0 that this class of inner products is naturally identified with $\mathbf{H}^2$. We thus have a bijection between Teichmüller space of the torus and $\mathbf{H}^2$.

It may be interesting to describe this same bijection in another way. Choose two simple closed curves on the torus, not homotopic to one another and none of them nullhomotopic; this gives us a basis of $H_1(\mathbf{T}^2)$. When we give an euclidean structure to the torus there will be closed geodesics, unique up to translation, homotopic to each of these curves: these are described by vectors in $\mathbf{R}^2$, or by complex numbers, well defined up to global similarities. We may therefore assume that the first curve corresponds to the complex number 1; the second complex number will be well defined and, assuming correct orientation, it will be on the upper half plane, which is, of course, one of our favorite models of $\mathbf{H}^2$. This defines the same bijection between Teichmüller space of the torus and $\mathbf{H}^2$ as we saw in the previous paragraph.

Notice that in Teichmüller space there are points corresponding to structures which are equivalent, but only through diffeomorphisms not isotopic to the identity. In our previous example (using the upper half plane model) the points i and $1 + i$ are equivalent through a diffeomorphism of the form $(x, y) \mapsto (x + y, y)$. In general, a diffeomorphism of M induces a map from Teichmüller space of M to itself; since isotopic diffeomorphisms induce the same map, we have an action of $\pi_0(\mathrm{Diff}(M))$ over Teichmüller space of M. Back to our example, we have an action of $SL(2, \mathbf{Z}) = \pi_0(\mathrm{Diff}(M))$ over $\mathbf{H}^2$, the Teichmüller space of $\mathbf{T}^2$. The reader can convince himself that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ takes z to $\frac{az+b}{cz+d}$; since $-\mathrm{Id}$ acts trivially, we may if we prefer say that it is $PSL(2, \mathbf{Z})$ which acts on $\mathbf{H}^2$. This action is a restriction of the usual action of $PSL(2, \mathbf{R})$ over $\mathbf{H}^2$ as its group of isometries.

The reader may be worried that we have been giving more structure to Teichmüller space than we have a right to. According to our definition, Teichmüller space has little more structure than a set; it is not very hard to show that it inherits a manifold structure from the class of conformal structures but that still does not justify making a distinction between Teichmüller space of the torus being $\mathbf{H}^2$ or, say, $\mathbf{R}^2$. As we shall see, it turns out that Teichmüller space does have a lot of natural extra structure: Teichmüller space has a natural metric and a natural complex analytic structure.

Let us begin by fixing a riemannian metric on M. We already saw that the space of metrics on a given point inducing the same area form is naturally identified with $\mathbf{H}^2$. We can identify a conformal structure with a metric with the same area form as the original one, that is, as a section of the bundle whose fibres are the previously mentioned copies of $\mathbf{H}^2$. It turns out that the structure of this bundle does not depend on the choice of the original metric. A point of Teichmüller space corresponds to an equivalence class of sections of this bundle; since each fibre has a natural metric, this defines a natural metric on Teichmüller space: the distance between two points is the minimum distance between sections representing these points where distance between sections is defined by the maximum (we have little option concerning this last issue since there is no given structure on M). This metric, which we shall not study here, is very important but it should be mentioned that it is *not* very nice.

Using the above construction, a tangent vector to Teichmüller space has as a representative a field of vectors tangent to each fibre defined along a section of the bundle which is a representative of the base point of the vector. There is a natural operation we can do with such a field of vectors: we can turn each vector counterclockwise by an angle of $\pi/2$ (we assume M oriented). It can be shown that this operation does not depend on choices of representatives, that is, it defines a linear transformation J with $J^2 = -\mathrm{Id}$ on each tangent plane to Teichmüller space. It turns out (but this is not at all obvious) that this defines a complex manifold structure on Teichmüller space.

Let us now look at the remaining examples of compact oriented surfaces M of dimension 2. A point in the Teichmüller space of M gives it a hyperbolic structure (up to equivalence by diffeomorphisms isotopic to the identity). Consider on M a set of disjoint simple closed curves that break up M into a union of "pairs of pants", i.e., surfaces diffeomorphic to the sphere minus three open balls with disjoint closures (Fig. 3.1). If M has genus g, we need $3g-3$ such curves. Given a hyperbolic structure, there is always a unique closed geodesic homotopic to each of non null-homotopic simple closed curve; this geodesic shall also be a simple closed curve. Our $3g-3$ curves therefore give us $3g-3$ invariants: the lengths of these geodesics. We shall from now on throw away the original curves and keep only the geodesics; each pair of pants has therefore as boundary three closed geodesics. It is easy to see that the lengths of the three geodesics determine the shape of the pair of pants and that, conversely, all lengths are permitted (constructing a pair of pants with prescribed boundaries is as easy as constructing a hexagon with all right angle given three alternate sides). The shape of the several pairs of pants does not yet define the shape of M: we have to decide how to glue them to one another. Notice that there are privileged points on the boundaries of pairs of pants: the points from which we can draw a common perpendicular to two boundaries. The relative position of such points tells us how to glue the pairs of pants, determining therefore the shape of M; on the other hand, they give us $3g-3$ more invariants (Fig 3.2). It follows that the Teichmüller space of M has real dimension $6g-6$, complex dimension $3g-3$. Teichmüller space is (real) diffeomorphic to $\mathbf{R}^{6g-6}$ but this does not respect the extra structure of Teichmüller space.

Just like in the case of the torus, if we twist one boundary of a pair of pants by 2π before glueing, we have an equivalent hyperbolic structure but a different point of Teichmüller space; this operation is called a Dehn twist. Thus, the parameter used to tell us how to glue pairs of pants was in $\mathbf{R}$, not S^1. Remember that the group $\pi_0(\mathrm{Diff}(M))$ acts on Teichmüller space of M: this shows we could use Teichmüller space in order to study diffeomorphisms of surfaces; we shall not follow this track here.

Let us conclude this part by talking very briefly on a related suject: laminations and train tracks. We shall always be working in a hyperbolic surface M.

A geodesic lamination of M is a closed subset Λ of M with the property that any intersection of a small neighborhood of a point of Λ with Λ is the union of disjoint geodesics in a unique way; this last condition guarantees that Λ has empty interior (Fig 3.2). The Gauss-Bonnet theorem shows that any lamination must have zero area. A lamination can be a finite union of simple, non-intersecting geodesics but it can also locally look like $K \times \mathbf{R}$, K being the Cantor set.

A train track is what you see when you look at a geodesic lamination if you do not have your glasses on. More precisely, a train track is a closed subset of M made up by finitely many smooth arcs which join each other in 'Y' shaped junctions, each side of the 'Y' being smooth (Fig 3.3). For technical reason we disallow polygons with one or two vertices (there is no point in building tracks which go to the same place).

A train track with weights is a train track where we assign to each segment a non-negative real number in such a way that at each junction the two incoming numbers add up to the outgoing number. Given a train track with weights there is a natural way to construct a unique lamination with a transversal measure: think of the weight on each arc as the amount of trafic along that edge; Thus, the weights tell geodesics which way to go.

Let PML (projective measured laminations) be the class of all geodesic laminations on M with a transversal measure, where we identify two measures which are one a constant multiple of the other. The class PML has a natural topology and it turns out that it is a manifold, large train tracks providing charts with their weights. It can be shown that this manifold is S^{6g-7} for M a hyperbolic surface of genus g.

The reader may by now suspect that $6g-6$ in the dimension of Teichmüller space and $6g-7$ in the dimension of PML have some relation to one another. It so happens that PML is in a natural sense the boundary of Teichmüller space; we shall not show how this identification works nor shall we prove anything here. We can nevertheless give a very vague idea of why this result makes sense. Think of taking a hyperbolic structure and deforming it: in some directions the metric became larger, in some it became smaller. If we keep deforming it till we go to 'infinity', along some direction the metric will have expanded very strongly while along another it will have contracted. The correct formalization of this imprecise idea of 'direction' is that of a measured lamination; we thus have a stable and an unstable measured laminations. We can also think that in the limit, the metric concentrates on the lamination, paying little attention (so to speak) to the rest of the manifold. These results are due to W. Thurston.

Part 4: Groups

We shall now look at discrete groups acting over manifolds, probably respecting some kind of structure, and at the resulting quotients. This subject is huge and this will force us to be superficial.

The definition of the quotient of a manifold by a group is very natural. Take the original space and identify points which are taken to one another by the action; in other words, an element of the quotient is an orbit of the action. The projection is a function from the manifold to the quotient which takes a point to its equivalent class. A subset of the quotient is open if and only if its inverse image by the projection is itself open.

The first thing we want from the quotient is that it should be a manifold. Let us consider an example. Let $\mathbf{Z}$ act on $\mathbf{R}^2$ as follows: $(x, y)^n = (2^n x, 2^{-n} y)$. Since the action is linear and preserves areas, we might hope for the quotient to be an affine manifold with area. It is easy to see that the fundamental group of the quotient is $\mathbf{Z}^2$: one of the generators was already there before the quotient (the unit circle) and another one was introduced by the quotient (a line from $(1,1)$ to $(2,1/2)$). You might think this quotient would be a torus but no, it is not compact. The answer is that the quotient is not Hausdorff.

This example shows the need to introduce definitions which will tell us when an action is nice enough that the quotient will be correspondingly nice. An action of a group G over a manifold M is called *wandering* if every point admits a neighborhood N such that $N \cap N^g \neq \emptyset$ for a finite number of $g \in G$ only. The action in the previous example is wandering. It is easy to see that if an action of G over N is wandering and free then the natural projection from M to M/G is a covering space. As we have seen, however, M/G need not be Hausdorff. We shall say that an action of G over a manifold M is *properly discontinuous* if for every compact set K, $K \subseteq M$, $K \cap K^g \neq \emptyset$ for finitely many $g \in G$ only. It is easy to show that if G acts in a free and properly discontinuous way over a Hausdorff, locally compact space M then M/G is Hausdorff. When the action is clear, we sometimes abuse notation and say that the group is wandering or properly discontinuous.

Let G be a discrete subgroup of isometries of euclidean or hyperbolic space. It turns out that G is properly discontinuous: the proof of this fact is not hard but we skip it. The consequence is that quotients by such groups are Hausdorff.

We have seen that the quotient of a manifold by a *free*, properly discontinuous action is a manifold. Many actions arise in practice which are not free. In such cases, the quotient will not be a manifold but a new kind of object, which we shall now study, called an *orbifold.*

An orbifold is defined in a way similar to a manifold: it is a topological space M equipped with a family of charts where each point has a neighborhood which is an image of a chart. The difference is, charts need not be homeomorphisms: they may cover the corresponding neighborhood of a point in the orbifold several times. More precisely, given an open subset A of $\mathbf{R}^n$ with a point x_0, a chart with domain A is a composition of a quotient by a finite group of diffeomorphisms of A leaving x_0 fixed with a homeomorphism.

Notice that the charts, and therefore also the above mentioned finite group, are part of the description of the orbifold: it is *not* enough to give the topological space. The condition concerning compatibility of charts needs to be translated to the new context. If $f : A \to U$ and $g : B \to V$ are charts and $U \cap V \neq \emptyset$ we can try to relate $f^{-1}(U \cap V)$ to $g^{-1}(U \cap V)$. We can not consider $g^{-1} \circ f$ since g is not necessarilly invertible. As we shall see in examples, there will in general not be a diffeomorphism between $f^{-1}(U \cap V)$ and $g^{-1}(U \cap V)$. Assuming $U \cap V$ connected and simply connected, we ask only for a diffeomorphism between a connected component A of $f^{-1}(U \cap V)$ and a connected component B of $g^{-1}(U \cap V)$ which makes the following diagram commute.

Let us look at examples of orbifolds. Here, pictures are probably more helpful than the text. An euclidean square whose sides are mirrors is an example of an orbifold (Fig. 4.0); in this example, the orbifold even comes with a geometric structure. A point in the interior of the square has regular charts; a point on one of the sides has a chart with a fold, so that the corresponding group is $\mathbf{Z}/2$, generated by a reflection; a vertex has charts that cover the neighborhood four times, with corresponding group $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ generated by two reflections along mutually perpendicular axis. It is easy to obtain this orbifold as a quotient of $\mathbf{R}^2$ or $\mathbf{T}^2$ by a discrete group of isometries, with properly discontinuous action.

Another interesting example is the sphere with four special points where the chart covers a neighborhood twice: here the group is $\mathbf{Z}/2$ generated by a rotation of π; such a point shall be called a point of order 2 (Fig. 4.1). This orbifold can also be obtained as a quotient of $\mathbf{R}^2$ or $\mathbf{T}^2$; the square of the previous paragraph is a quotient of this example by $\mathbf{Z}/2$.

Other examples are spheres with special points. If we have three special points of orders 2, 3 and 5, the orbifold is the quotient of the sphere by a finite group (Fig. 4.2). If they have orders 2, 3 and 6, the orbifold is a quotient of the plane or the torus (Fig. 4.3). If the orders of the three points are 2, 3 and 7, the orbifold is a quotient of the hyperbolic plane by a group of isometries (Fig. 4.4). If, however, the sphere has a single special point of any order (greater than 1) we have an example of an orbifold which can not be constructed as a quotient of a manifold by a discrete group (Fig. 4.5).

In order to define Euler characteristic of an orbifold, let us make a cell decomposition of the orbifold as we would do for a manifold. The square whose sides are mirrors, for instance, is decomposed as one 2-cell, four 1-cells (the sides) and four 0-cells (the vertices). How should these be counted for the Euler characteristic? The 2-cell is perfectly sane but the 1-cells are *not*: they are mirrors, and if we take a double cover of a small neighborhood of one of them, they give rise, not to two, but to one 1-cell. It is appropriate therefore that each of these should count only as *half* a 1-cell when we compute the Euler characteristic. Similarly, each vertex should count only as one quarter of a 0-cell. The Euler characteristic of this example is therefore $1 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 0$.

Proceeding to the other examples, a sphere with three special points can be decomposed into one 2-cell, two 1-cells (joining special points) and three 0-cells; the 2-cell and 1-cells are regular but the 0-cells are not: each one should count as $1/n$, where n is the order of the point. The Euler characteristic for our three examples are therefore $1 - 1 - 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{1}{30}$, $1 - 1 - 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$ and $1 - 1 - 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = -\frac{1}{42}$. Finally, the sphere with one point of order n has Euler characteristic $1 + 1/n$.

Let us define the notion of a covering space of an orbifold. A covering space is a function π from an orbifold M to another orbifold N such that, for every point $x \in N$, there exists an open neighborhood V of x satisfying the properties below.

- The open set V must be the image of one of the charts defining the orbifold structure of N; let $f : A \to N$ be this chart.
- The inverse image $\pi^{-1}(V)$ must be the disjoint union of connected components U_i, U_i open subsets of M, such that for each U_i there is a chart $g_i : A \to U_i$ (this is the same A we saw before) with $f = \pi \circ g$.

This definition generalizes the usual definition which applies to manifolds. Notice that the Euler characteristic behaves for orbifolds with respect to covering spaces exactly as it does for manifolds: if M covers N with multiplicity k the Euler characteristic of M is equal to k times the Euler characteristic of N. The multiplicity is defined as the number of preimages under π of a *regular* point of N.

All previously mentioned quotients are covering spaces. The sphere with four points of order 2 is a double cover of the square whose sides are mirrors; more generally, given any orbifold with mirrors we can take a double cover by taking two copies of the orbifold and glueing them to one another along the mirrors. We are now ready to make sense of our conclusion relating all our concepts: given a properly discontinuous action of G over M, the quotient is an orbifold and the projection a covering space.

As a corollary of the definition of covering space we get the notions of universal cover, fundamental group and simply connected orbifold. Let us give a more explicit construction of the fundamental group via generators and relations. If the orbifold has any mirrors, take the double cover as previously described and compute the fundamental group of the covering orbifold: the fundamental group of the original orbifold will be an extension of the fundamental group of the covering orbifold by $\mathbf{Z}^2$. Having done this, the set of special points has codimension at least 2. The set of regular points is therefore a connected manifold (we assume the orbifold is connected). Compute the fundamental group of this manifold: the fundamental group of the orbifold is obtained by adding up some relations, that is, by taking an appropriate quotient. Notice that only parts of the special set with codimension 2 are relevant; in this situation the group of diffeomorphisms defining the orbifold structure must be a finite cyclic group of rotations around a subspace of codimension 2. We must therefore add relations of the form $g^n = e$ where g is the element of the fundamental group of the manifold corresponding to going once around the relevant special cell, n is the order of the cyclic group of rotations and e is identity.

Let us see some examples. The sphere with four special points of order 2 has fundamental group generated by a, b and c with relations $a^2 = b^2 = c^2 = abcabc = e$. We already know the answer here since we already saw a covering of this orbifold by the plane. An explicit construction of this group as a subgroup of orientation preserving isometries of the plane is: let a, b and c all be rotations by π around the points $(0,0)$, $(1,0)$ and $(0,1)$, respectively (any three non-colinear points will do). In this example we used known geometric information to find out what a presentation of a group really meant; in the next example we do the opposite. Consider the sphere with two special points of orders n and m (notice that $m = 1$ gives the case of one special point only). The construction of the fundamental group gives one generator a and relations $a^n = a^m = e$; it is easy to see that this corresponds to $\mathbf{Z}/\gcd(n,m)$ In particular, if $\gcd(n,m) = 1$ the orbifold is simply connected and admits no non-trivial cover. This is an example of an orbifold which is not the quotient of a manifold.

We have seen four kinds of orbifolds of dimension 2: some admit hyperbolic structures, some euclidean, some spherical and finally some admit no reasonable structure whatsoever since they do not even admit a covering by a manifold. This classification does indeed hold for all orbifolds of dimension 2, and, as in the case of manifolds, all but a finite number are hyperbolic. We shall present this classification here for the case of compact orientable orbifolds without boundary. The only such orbifold which admit no covering by manifolds are spheres with one special point or two special points of different orders. The spherical examples are spheres with two special points of the same order or three special points of orders 2,2,n or 2,3,3 or 2,3,4 or 2,3,5. The euclidean examples are spheres with three points of orders 2,3,6 or 2,4,4 or 3,3,3 ou with four special points of orders 2,2,2,2. All other compact orientable orbifolds without boundary of dimension 2 are hyperbolic.

Let us finish our presentation of orbifolds with a look at orbifolds of dimension 3. What kinds of special points can we have? In codimension 1 we can have mirrors: boundary surfaces where the group giving the structure of the orbifold is $\mathbf{Z}/2$ generated by a reflection. In codimension 2 we have curves of special points, and these come in two flavors. We can have isolated curves where the group giving the structure of the orbifold is $\mathbf{Z}/n$ made up with rotations. We can also have a curve between two mirrors where our group is D_n, a dihedral group generated by two reflections in different planes. Finally, in codimension 1 we have special points. In order to understand these we have to look back to spherical orbifolds of dimension 2: a neighborhood of the point is like the tip of a cone over the orbifold of lower dimension. The sphere corresponds to regular points. The projective plane corresponds to the only possible kind of isolated special point: the tip of a cone over the projective plane. The group giving the structure of the orbifold in this case is $\mathbf{Z}/2$ generated by the antipodal map in $\mathbf{R}^3$. A sphere with two special points of the same order corresponds to a point on a curve of that order. A disk whose boundary is a mirror corresponds to a point on a mirror face. A bigon whose sides are mirrors and with vertices of the same order corresponds to a point on a curve between two mirrors. A sphere with three special points corresponds to a point where three curves meet: remember that the orders of the points (and therefore of the curves) must be 2,2,n or 2,3,3 or 2,3,4 or 2,3,5. Finally, a triangle whose sides are mirrors corresponds to a point where three mirrors meet: here too there are restrictions on the orders of the incoming curves. We can build an impressive amount of different examples: take into account that the base set can be any 3-manifold (more than that, since tips of projective plane cones are allowed) and that the special curves can be knotted.

There is an important issue which we have been ignoring. In the definition of orbifolds, we have considered a finite subgroup of diffeomorphisms of $\mathbf{R}^n$ without any additional structure. Later, however, we put a riemannian metric on the orbifold. This can only work if the action of our original finite group of diffeomorphisms over $\mathbf{R}^n$ is isomorphic to a subgroup of $SO(n, \mathbf{R})$. Is this always the case? Yes, it is enough to start with any metric and take averages using the finite group. We have a much harder question if we start with a finite group of diffeomorphisms of S^{n-1}.

Let us now take a quick look at a vast subject: that of Fuchsian and Kleinian groups. A Fuchsian group is a discrete group of isometries of $\mathbf{H}^2$, that is, a discrete subgroup of $PSL(2, \mathbf{R})$. A Kleinian group is a discrete group of isometries of $\mathbf{H}^3$, that is, a discrete subgroup of $PSL(2, \mathbf{C})$.

The first thing to do in order to understand such groups is to understand its elements: let us classify, up to conjugation, isometries of $\mathbf{H}^2$ and $\mathbf{H}^3$. As we shall see, isometries of $\mathbf{H}^2$ are naturally classified as elliptic, parabolic or hyperbolic.

Since we identify the group of isometries of $\mathbf{H}^2$ with $PSL(2, \mathbf{R})$, to each isometry A corresponds a number $\operatorname{tr}(A)$ which is well defined up to sign, or, more practically, we have a well defined value of $\operatorname{tr}^2(A)$. Linear algebra now tells us that, since all determinants are equal to 1, the trace defines the matrix up to conjugation *except* if it is equal to 2 (or -2). In $PSL(2, \mathbf{R})$, $\operatorname{tr}^2(A)$ determines a conjugacy class in $PGL(2, \mathbf{R})$ of A except for $\operatorname{tr}^2(A) = 4$. Notice that a conjugacy class of $PGL(2, \mathbf{R})$ which is contained in $PSL(2, \mathbf{R})$ may split into more than one conjugacy class of $PSL(2, \mathbf{R})$. In case $\operatorname{tr}^2(A) = 4$, there are two possibilities: A can either be the identity or can be conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The case $0 \leq \operatorname{tr}^2(A) < 4$ is called elliptical and corresponds to matrices of the form $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$; in hyperbolic plane this corresponds to a rotation by an angle t around some point. Notice that rotations by t and $-t$ are conjugate only in the group of all isometries but not in the group of orientation preserving isometries. Such an isometry has no fixed point on the circle at infinity which corresponds to the fact that the matrix has no real eigenvalues or eigenvectors.

The case $\operatorname{tr}^2(A) = 4$, $A \neq \mathrm{Id}$, is called parabolic and corresponds to matrices in $PSL(2, \mathbf{R})$ conjugate to either $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$; in $PSL(2, \mathbf{R})$ these matrices are not conjugate. In order to understand the geometrical meaning of these matrices it is best to look at the upper half plane model, where they correspond to the functions $f(z) = z+1$ and $g(z) = z-1$. This kind of isometry has exactly one fixed point on the circle at infinity, corresponding to the unique eigenvector (up to scalar factor) of the matrix.

The case $\operatorname{tr}^2(A) > 4$ is called hyperbolic and corresponds to matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ or, equivalently, $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$. This represents a translation by t in hyperbolic plane; remember that a translation in $\mathbf{H}^2$ is performed along a well defined axis (a geodesic). Such isometries have two fixed points on the circle at infinity: the two extremes of the translation axis. These also correspond to the two eigenvectors of the matrix.

In $PSL(2, \mathbf{C})$ we can still compute $\operatorname{tr}^2(A)$; if this number is a non-negative real, the isometry fixes a plane and on that plane the previous discussion holds; notice that now a rotation by t and a rotation by $-t$ are conjugate and so are all parabolic isometries. If the square of the trace is not a non-negative real number the isometry corresponds to a translation followed by a rotation with the same axis; in particular, if this is a negative real number the rotation is by π. We now always have two fixed points on the sphere at infinity except in the parabolic case, where we still have only one.

In part 0 we already saw examples of Fuchsian groups coming from tilings of $\mathbf{H}^2$. Notice that a free group with two generators is Fuchsian: just take in $\mathbf{H}^2$ a degenerate square (i.e., with vertices on the circle at infinity) and consider the group generated by translations taking one side to the opposite one. As a consequence, free groups with any finite or countably infinite number of generators are all Fuchsian since these are subgroups of the free group with two generators; alternatively, we can come up with a geometric description.

Let us show that $\mathbf{Z}^2$ is not Fuchsian. Suppose by contradiction that we have a copy of $\mathbf{Z}^2$ as a group of isometries of $\mathbf{H}^2$. Clearly this group has no elliptical elements, since these have to have finite order if they are to be in a discrete group. The quotient of $\mathbf{H}^2$ by our group is therefore a manifold of dimension 2, with fundamental group equal to $\mathbf{Z}^2$ and a hyperbolic structure. It turns out, however, that the only manifold of dimension 2 whose fundamental group is $\mathbf{Z}^2$ is the torus, and we know (by Gauss-Bonnet) that the torus does not admit hyperbolic structure.

We claimed in part 0 (with no proof) that for any tiling of $\mathbf{H}^2$ the tiles have area at least $\pi/42$ (provided the group of symmetries acts transitively on tiles). This can be shown by looking at possible values for the Euler characteristic of orbifolds: it is not hard to see that the smallest possible absolute value for a negative Euler characteristic of an orbifold is $-1/84$ for a triangle whose sides are mirrors and whose vertices have orders 2, 3 and 7. We want, however, to consider another point of view relating this fact and the observation in the previous paragraph to the following theorem concerning Kleinian groups known as the Jørgensen inequality.

Remember that a Kleinian group acts on $\mathbf{D}^3$, the closed unit ball in $\mathbf{R}^3$, as $\mathbf{H}^3$ plus the sphere at infinity. We say a Kleinian group is *elementary* if any orbit of this action is finite; elementary groups are more or less trivial and non-interesting (we shall not classify them here). Jørgensen inequality states the following: if f e g generate a non-elementary Kleinian group then

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}(fgf^{-1}g^{-1}) - 2| \geq 1.$$

Intuitively, this says that f e g can not be both near the identity. We shall not prove this theorem: let us only say that the fact that $\mathbf{Z}^2$ is not Fuchsian, the estimate on the minimum area of a tile and other similar results are consequences of this theorem.

Let us finish our very brief survey on Fuchsian and Kleinian groups talking about the limit set and the Ahlfors measure conjecture. Given a Kleinian group, consider some orbit of its action on $\mathbf{H}^3$. This is a closed set in $\mathbf{H}^3$ but not in $\mathbf{D}^3$, the union of $\mathbf{H}^3$ with the sphere at infinity. The limit set of the group is the part of the closure of the orbit which lies in the sphere at infinity; this set does not depend on the choice of the orbit. It is also the smallest non-empty compact subset of $\mathbf{S}^2$ which is invariant under the action of the group.

For a Fuchsian group, the limit set can be made up of one or two points (these are extreme, non-interesting cases), a Cantor set or S^1 itself. In the case of Kleinian groups there are many more possibilities, which we shall not discuss. Let us merely state what is probably the most important open question in this field, the Ahlfors measure conjecture: if a limit set is not the whole sphere, it has zero Lebesgue measure.

Part 5: Geometric structures on manifolds of dimension 3

In the previous parts we saw results concerning structures on manifolds of dimension 2; at the end, we had a reasonably complete picture. We now look at geometric structures on manifolds of dimension 3. This situation is incredibly more complex, and less well known, than the previous one: remember that even the understanding of 3-manifolds, no structures involved, is formidable if at all possible. Geometric structures did actually help us here, and we may hope that through them one day 3-manifolds will be better understood.

The first question is: what kind of structures are we interested in? We already know three examples: hyperbolic, euclidean and spherical geometry. It turns out that, even though this was enough in dimension 2, it is not at all enough in dimension 3: we have to introduce new kinds of geometries. We expect them to have a transitive group of isometries; we are not interested in trivial changes of scale; finally, there should exist discrete subgroups of the group of isometries such that the quotient is a compact manifold. It turns out that this leaves us with five new geometries which we shall now describe.

By taking the cartesian product with a line, we get $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$. The isometries of such spaces must treat each factor separately (because of different sectional curvatures).

A sixth kind of geometry is given by the unit tangent bundle to $\mathbf{H}^2$; notice that there is a natural diffeomorphism (up to choice of a base point) between this manifold and

$PSL(2,\mathbf{R})$ but the first description is more suggestive of the correct metric. This manifold is not simply connected (it is homotopic to S^1) so it is convenient to take as model space the universal cover. The isometries are generated by those coming from isometries of $\mathbf{H}^2$ and transformations turning each vector by t, keeping all base points fixed.

A seventh kind of geometry, which we call Nil, is that of the nilpotent group of matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us give another equivalent description by giving a metric in $\mathbf{R}^3$ defined by an orthonormal basis for the tangent plane at each point; at the point (x,y,z) the orthonormal basis shall be $(1,0,0)$, $(0,1,x)$ and $(0,0,1)$.

Finally, for our eighth geometry, which we call Sol, we begin by giving a model as $\mathbf{R}^3$ with a different metric again given by an orthonormal basis at each point. At the point (x,y,z), this shall be $(e^z,0,0)$, $(0,e^{-z},0)$ and $(0,0,1)$. The name of this geometry comes from the fact that its group of isometries is solvable; it is sometimes convenient to think of it as the geometry of this solvable group.

Let us suggest an alternative way to understand these last two geometries based on discretizations. More precisely, we shall look at Cayley diagrams for certain discrete groups where all edges should be thought of as having length 1; these look like distorted lattices ($\mathbf{Z}^3$) and they illustrate our geometries. Actually, $\mathbf{Z}^3$ is the analogue of these constructions for euclidean geometry, where we connect (x,y,z) to $(x\pm 1,y,z)$, $(x,y\pm 1,z)$ and $(x,y,z\pm 1)$: this is the Cayley diagram of the group with generators a, b and c and relations $[a,b]=[a,c]=[b,c]=e$.

For Nil, we take $\mathbf{Z}^3$ as the set of points but we connect (x,y,z) to $(x\pm 1,y,z)$, $(x,y\pm 1,z\pm x)$ e $(x,y,z\pm 1)$. Thus in the vertical direction nothing changes but if we draw a closed curve on the xy plane and lift it to this geometry the z coordinate comes back increased by the surrounded area (with sign). This corresponds to a group N with generators a, b and c and relations $[a,b]=c$, $[a,c]=[b,c]=e$; notice that this group is nilpotent and can be factored by an exact sequence of the form

$$0\to\mathbf{Z}\to N\to\mathbf{Z}^2\to 0,$$

the first $\mathbf{Z}$ being the centre of N.

As to Sol, we again take $\mathbf{Z}^3$ for the set of points and connect (x,y,z) to $(x\pm 1,y,z)$, $(x,y\pm 1,z)$, $(2x+y,x+y,z-1)$ and $(x-y,-x+2y,z+1)$. This corresponds to the group S with generators a, b e c and relations $[a,b]=e$, $a^c=a^2b$ e $b^c=ab$; this group is solvable (but not nilpotent) and can be factored by an exact sequence of the form

$$0\to\mathbf{Z}^2\to S\to\mathbf{Z}\to 0,$$

where the first $\mathbf{Z}^2$ is the group generated by commutators, i.e., the subgroup generated by a and b. An equivalent picture is: connect (x, y, z) to $(x \pm a(z), y \pm b(z), z)$, $(x \pm c(z), y \pm d(z), z)$ and $(x, y, z \pm 1)$ where

$$\begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^z.$$

The choice of the matrix on the right hand side is more or less arbitrary; any matrix in $SL(2, \mathbf{Z})$ with simple real positive spectrum will do.

Let us give some examples of manifolds with geometric structures. We got many examples of surfaces by glueing sides of polygons. If we glue faces of polyhedra instead (in the right way) we get 3-manifolds with geometric structures. As an example (due to Poincaré) take a dodecahedron and identify opposite faces. There are two natural and quite different ways of doing this: we may glue the opposite faces after turning $\pi/5$ or after turning $3\pi/5$. (A rotation by π would give us $\mathbf{P}^3$ but that ignores the combinatorics of the dodecahedron.)

With the help of drawings, the reader can check that in the $3\pi/5$ case all vertices are identified and that the pattern around this vertex is a icosahedron. The geometric structure at the vertex requires that the angles at the vertices of the dodecahedron have a certain given value, which we shall not compute but which is clearly less than what we have for an euclidean dodecahedron. If we take a regular dodecahedron in hyperbolic space the angles shall be smaller than in the euclidean case. For very small dodecahedra the angles tend to the euclidean value and for very large dodecahedra the angles tend to zero. By adjusting the size of the dodecahedron we can therefore find one with the correct angles. It is therefore possible to endow the manifold with a hyperbolic structure.

In the case $\pi/5$, however, vertices are identified in five sets of four vertices each and the picture at a vertex is a tetrahedron. Unlike the previous example, the needed angle is now more than in the euclidean case and now we can construct our manifold giving it a spherical structure. If we look at the universal cover, we have the decomposition of $\mathbf{S}^3$ into 120 dodecahedra corresponding to the regular polytope called $\{5, 3, 3\}$ in Schläfli's notation. There is another way to think of this example. Start with $\mathbf{S}^3$ with the group structure given by quaternions or equivalently as a double cover of $SO(3, \mathbf{R})$. Now, $\mathbf{S}^3$ contains a discrete group of order 120: the lifting of the group of symmetries of the dodecahedron, which is a copy of A_5 inside $SO(3, \mathbf{R})$. Our manifold is the quotient of $\mathbf{S}^3$ by this group. This is an example of a manifold with the same homology groups as $\mathbf{S}^3$ but which is not homeomorphic to $\mathbf{S}^3$ since it is not simply connected.

In dimension 2 every manifold admits a complete geometric structure. The same does not hold in dimension 3. There is however hope that, after a suitable decomposition, every 3-manifold has a geometric structure: this is Thurston's geometrization conjecture.

Let us explain what the decomposition mentioned above is. We all know how to define the connected sum of two surfaces: remove a small disc from each surface and glue the resulting boundaries to one another. The generalization of this idea to dimension 3 is natural: remove a small ball from each manifold and glue the resulting (spherical) boundaries to one another. It is not so easy as it may seem to verify that this definition is healthy (there are big problems in higher dimensions) but it turns out that this works: connected sum is well defined (but you have to pay attention to orientation when you glue) and any 3-manifold can be decomposed in an essentially unique way as a connected sum of irreducible 3-manifolds.

This decomposition is not sufficient for the geometrization conjecture. Cutting along spheres is not enough: it is also necessary to cut along tori. Again, the idea is intuitively clear, and despite non-trivial technical issues, the decomposition does after all work well. The geometrization conjecture claims that whatever is left after performing these two decompositions should be geometrizable with one of our eight geometries.

A corollary of the geometrization conjecture is the Poincaré conjecture in dimension 3: any compact, connected, simply connected 3-manifold is homeomorphic to $\mathbf{S}^3$. Poincaré himself actually claimed that this was true, and seemed to be convinced that it was more or less trivial, but shortly afterwards realized that this was a hard question. And hard indeed it seems to be for it has resisted all attempts to prove it. By the way, this question was generalized to other dimensions in the following form: if M is a compact, connected manifold of dimension n with $\pi_k(M)$ trivial for $k < n$ then M is homeomorphic to $\mathbf{S}^n$. By now, it has been shown that the conjecture holds for all $n > 3$, ($n > 4$ is due to Smale and $n = 4$ to Freedman) but the original case $n = 3$ resists.

Let us see how Thurston's conjecture implies Poincaré's. Take any compact, connected, simply connected 3-manifold M and suppose by contradiction that it is not S^3. By performing decompositions we eventually get an irreducible and therefore, by Thurston's conjecture, geometrizable, compact, connected, simply connected 3-manifold N distinct from S^3 (S^3 is a neutral element for connected sum). Since N is simply connected, the geometric structure is a diffeomorphism between N and one of the eight models; this is absurd since the only compact model is S^3.

The geometrization conjecture is far from settled but many strong partial results have been obtained. The main one is certainly Thurston's result that all 3-manifolds satisfying certain hypothesis are hyperbolic; the hypothesis are such that most 3-manifolds satisfy them. This is a very hard theorem: we shall not even state it precisely here, much less consider the proof.

References:

[A] Ahlfors, Lars V., *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill Book Company.

[B] Beardon, Alan F., *The Geometry of Discrete Groups*, Springer-Verlag (GTM 91).

[CB] Casson, Andrew J.;Bleiler, Steven A., *Automorphisms of Surfaces after Nielsen and Thurston*, Cambridge University Press, London Mathematical Society Student Texts 9.

[D] Dehn, Max, *Papers on Group Theory and Topology*, Springer-Verlag.

[E0] Epstein, D.B.A.(editor), *Analytic and Geometric Aspects of Hyperbolic Space*, Cambridge University Press, London Mathematical Society, Lecture Notes Series, 111.

[E1] Epstein, D.B.A.(editor), *Low-dimensional Topology and Kleinian Groups* Cambridge University Press, London Mathematical Society, Lecture Notes Series, 112.

[L] Lehto, Olli, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag (GTM 109).

[N] Nag, Subhashis, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley-Interscience.

[P] Poincaré, Henri*Papers in Fuchsian Functions*, Springer-Verlag.

[T0] Thurston, William P., *The Geometry and Topology of Three-Manifolds*, preprint.

[T1] Thurston, William P., *Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry*, Bull. of the A. M. S., Vol. 6, No. 3, May 1982.

[T2] Thurston, William P., *Hyperbolic structures on 3-manifolds, I, II, III*; Annals of Mathematics, 124 (1986), 203-246; preprints.

Nicolau C. Saldanha
Departamento de Matemática, PUC-Rio
Rua Marquês de São Vicente 225
Gávea, Rio de Janeiro
RJ 22453-900, Brasil

nicolau@mat.puc-rio.br
http://www.mat.puc-rio.br/∼nicolau/