The homotopy and cohomology of spaces of locally convex curves in the sphere — I

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Abstract

A smooth curve $\gamma : [0,1] \to \mathbb{S}^2$ is locally convex if its geodesic curvature is positive at every point. J. A. Little showed that the space of all locally positive curves γ with $\gamma(0) = \gamma(1) = e_1$ and $\gamma'(0) = \gamma'(1) = e_2$ has three connected components $\mathcal{L}_{-1,c}$, $\mathcal{L}_{+1}$, $\mathcal{L}_{-1,n}$. The space $\mathcal{L}_{-1,c}$ is known to be contractible but the topology of the other two connected components is not well understood. We study the homotopy and cohomology of these spaces. In particular, for $\mathcal{L}_{-1} = \mathcal{L}_{-1,c} \sqcup \mathcal{L}_{-1,n}$, we show that $\dim H^{2k}(\mathcal{L}_{(-1)^k}, \mathbb{R}) \geq 1$, that $\dim H^{2k}(\mathcal{L}_{(-1)^{(k+1)}}, \mathbb{R}) \geq 2$, that $\pi_2(\mathcal{L}_{+1})$ contains a copy of $\mathbb{Z}^2$ and that $\pi_{2k}(\mathcal{L}_{(-1)^{(k+1)}})$ contains a copy of $\mathbb{Z}$.

1 Introduction

A curve $\gamma : [0,1] \to \mathbb{S}^2$ is called *locally convex* if its geodesic curvature is always positive, or, equivalently, if $\det(\gamma(t), \gamma'(t), \gamma''(t)) > 0$ for all t. Let $\mathcal{L}_I$ be the space of all locally convex curves γ with $\gamma(0) = \gamma(1) = e_1$ and $\gamma'(0) = \gamma'(1) = e_2$. The topology in this space of curves will be given by the Sobolev metric H^2: this has the minor technical advantages of making $\mathcal{L}_I$ a Hilbert manifold and of allowing for jump discontinuities in γ'' in the constructions. The choice of metric actually makes very little difference: since it is easy to uniformly smoothen out a curve while keeping its geodesic curvature positive we might just as well work with the C^∞ topology, or with C^k for some $k \geq 2$.

J. A. Little [7] showed that $\mathcal{L}_I$ has three connected components $\mathcal{L}_{-1,c}$, $\mathcal{L}_{+1}$, $\mathcal{L}_{-1,n}$: we call these the Little spaces. Figure 1 shows examples of curves in

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$\mathcal{L}_{-1,c}$, $\mathcal{L}_{+1}$ and $\mathcal{L}_{-1,n}$, respectively. The connected component $\mathcal{L}_{-1,c}$ consists of the simple curves in $\mathcal{L}_I$: the space $\mathcal{L}_{-1,c}$ is known to be contractible ([15], Lemma 5). The topology of these and related spaces has been discussed, among others, by B. Shapiro, M. Shapiro and Khesin ([14], [13]) but the topology of the Little spaces is still not well understood. This series of papers is to present new results concerning the homotopy and cohomology of the Little spaces. A more ambitious aim would be to determine the homotopy type of these spaces (which we hope to accomplish in [9]).

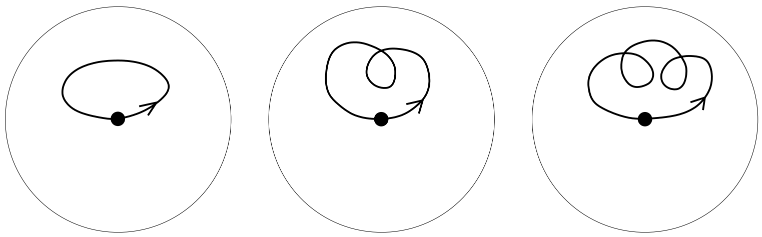

Figure 1: Curves in $\mathcal{L}_{-1,c}$, $\mathcal{L}_{+1}$ and $\mathcal{L}_{-1,n}$.

Let $\mathcal{I}_I \supset \mathcal{L}_I$ be the space of immersions $\gamma : [0,1] \to \mathbb{S}^2$, $\gamma'(t) \neq 0$, $\gamma(0) = \gamma(1) = e_1$, $\gamma'(0) = \gamma'(1) = e_2$. For each $\gamma \in \mathcal{I}_I$, consider its Frenet frame $\mathfrak{F}_\gamma : [0,1] \to SO(3)$ defined by

$$\begin{pmatrix} \gamma(t) & \gamma'(t) & \gamma''(t) \end{pmatrix} = \mathfrak{F}_\gamma(t) R(t),$$

$R(t)$ being an upper triangular matrix with positive diagonal (the left hand side is the 3×3 matrix with columns $\gamma(t)$, $\gamma'(t)$ and $\gamma''(t)$). The universal (double) cover of $SO(3)$ is $\mathbb{S}^3 \subset \mathbb{H}$, the group of quaternions of absolute value 1; let $\Pi : \mathbb{S}^3 \to SO(3)$ be the canonical projection. The curve $\mathfrak{F}_\gamma$ can be lifted to define $\tilde{\mathfrak{F}}_\gamma : [0,1] \to \mathbb{S}^3$ with $\tilde{\mathfrak{F}}_\gamma(0) = 1$, $\Pi \circ \tilde{\mathfrak{F}}_\gamma = \mathfrak{F}_\gamma$. The value of $\tilde{\mathfrak{F}}_\gamma(1)$ defines the two connected components $\mathcal{I}_{\pm 1}$ of $\mathcal{I}_I$: $\gamma \in \mathcal{I}_{+1}$ if and only if $\tilde{\mathfrak{F}}_\gamma(1) = 1$. Notice that if γ is a simple curve in $\mathcal{I}_I$ then $\tilde{\mathfrak{F}}_\gamma(1) = -1$ and therefore $\gamma \in \mathcal{I}_{-1}$. We have $\mathcal{L}_{+1} = \mathcal{I}_{+1} \cap \mathcal{L}_I$ and $\mathcal{L}_{-1} = \mathcal{L}_{-1.c} \sqcup \mathcal{L}_{-1,n} = \mathcal{I}_{-1} \cap \mathcal{L}_I$.

Let $\Omega\mathbb{S}^3$ (resp. $\Omega_{-}\mathbb{S}^3$) be the set of continuous curves $\alpha : [0,1] \to \mathbb{S}^3$, $\alpha(0) = \alpha(1) = 1$ (resp. $\alpha(0) = 1$, $\alpha(1) = -1$). These two spaces are easily seen to be homeomorphic and shall from now on be identified; $\Omega\mathbb{S}^3$ is a well understood space: we have $H^*(\Omega\mathbb{S}^3, \mathbb{R}) = \mathbb{R}[\mathbf{x}]$ where $\mathbf{x} \in H^2$ satisfies $\mathbf{x}^n \neq 0$ for all positive n (see, for instance, [1]). The previous paragraph defines maps $\tilde{\mathfrak{F}} : \mathcal{I}_{\pm 1} \to \Omega\mathbb{S}^3$. It is a well-known fact that these two maps are homotopy equivalences; this follows from the Hirsch-Smale Theorem ([8], [6], [16]). As we shall see, the inclusions $i : \mathcal{L}_{\pm 1} \to \mathcal{I}_{\pm 1}$ are not homotopy equivalences.

Theorem 1 *For any compact space K and any function $f : K \to \mathcal{I}_{\pm 1}$ there exists $g : K \to \mathcal{L}_{\pm 1}$ and a homotopy $H : [0,1] \times K \to \mathcal{I}_{\pm 1}$ with $H(0,k) = f(k)$ and $H(1,k) = g(k)$ for all $k \in K$.*

The maps $i : \mathcal{L}_{\pm 1} \to \mathcal{I}_{\pm 1}$ therefore induce surjective maps $\pi_k(\mathcal{L}_{\pm 1}) \to \pi_k(\mathcal{I}_{\pm 1})$. In fact, since $\mathcal{L}_{\pm 1}$ and $\mathcal{I}_{\pm 1}$ have the homotopy type of CW complexes with finitely many cells per dimension, the inclusions are homotopically surjective but we skip the details.

In particular, $\mathcal{L}_{\pm 1}$ is not homotopically equivalent to a finite CW-complex. Theorem 2 in [11] is a similar result for arbitrary dimension. From Theorem 1 we write $\mathbb{R}[\mathbf{x}] \subseteq H^*(\mathcal{L}_{\pm 1}; \mathbb{R})$. The main result of this paper implies that $\mathbb{R}[\mathbf{x}]$ has infinite codimension as a subspace of $H^*(\mathcal{L}_{\pm 1})$.

Let $\mathcal{L}$ be the contractible space of all locally convex curves γ with $\gamma(0) = e_1$ and $\gamma'(0) = e_2$ and define $\phi : \mathcal{L} \to \mathbb{S}^3$ by $\phi(\gamma) = \tilde{\mathfrak{F}}_\gamma(1)$ where $\tilde{\mathfrak{F}}_\gamma : [0,1] \to \mathbb{S}^3$ is defined via Frenet frames as above. It is natural to conjecture that $\phi : \mathcal{L} \to \mathbb{S}^3$ is somehow similar to a fibration.

This is not the case: we prove that the map ϕ does not satisfy the homotopy lifting property. Let $X = [0,1]^2$. There exist maps $\mathbf{h} : X \times [0,1] \to \mathbb{S}^3$ and $\tilde{\mathbf{h}}_0 : X \times \{0\} \to \mathcal{L}$ such that $\phi \circ \tilde{\mathbf{h}}_0 = \mathbf{h}|_{X \times \{0\}}$ and there exists no map $\tilde{\mathbf{h}} : X \times [0,1] \to \mathcal{L}$ with $\phi \circ \tilde{\mathbf{h}} = \mathbf{h}$. The maps $\mathbf{h}$ and $\tilde{\mathbf{h}}_0$ will be constructed explicitly in Section 6.

A curve $\gamma \in \mathcal{L}_{(-1)^{(k+1)}}$ is a *flower* of $2k+1$ petals if there exist $0 = t_0 < t_1 < t_2 < \cdots < t_{2k} < t_{2k+1} = 1$ and $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{2k} < \theta_{2k+1} = \theta_M$ such that:

1. $\gamma(t_1) = \gamma(t_2) = \cdots = \gamma(t_{2k}) = e_1$;
2. the only self-intersections of the curve γ are of the form $\gamma(t_i) = \gamma(t_j)$;
3. the argument of $(x_{i,2}, x_{i,3})$ is θ_i, where $(0, x_{i,2}, x_{i,3}) = (-1)^i \gamma'(t_i)$.

As a somewhat degenerate case, a flower of 1 petal is a simple locally convex curve. Figure 2 shows examples of flowers.

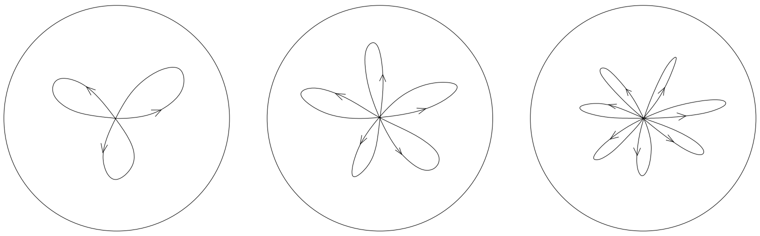

Figure 2: Examples of flowers with 3, 5 and 7 petals.

For $k > 0$, let $\mathcal{F}_{2k} \subset \mathcal{L}_{(-1)^{(k+1)}}$ be the set of flowers of $2k+1$ petals. As we shall see in Lemma 4.1, the set $\mathcal{F}_{2k}$ is closed and a submanifold of codimension $2k$.

Furthermore, the normal bundle to $\mathcal{F}_{2k}$ in $\mathcal{L}_{(-1)^{(k+1)}}$ is trivial. Thus, intersection with $\mathcal{F}_{2k}$ defines an element $\mathbf{f}_{2k} \in H^{2k}(\mathcal{L}_{(-1)^{(k+1)}})$ with $(\mathbf{f}_{2k})^2 = 0$.

We shall construct maps $\mathbf{g}_{2k} : \mathbb{S}^{2k} \to \mathcal{L}_{(-1)^{(k+1)}}$ which are homotopic to a constant in $\mathcal{I}_{(-1)^{(k+1)}}$ but which satisfy $\mathbf{f}_{2k}\mathbf{g}_{2k} = 1$, thus proving that both $\mathbf{f}_{2k} \in H^{2k}(\mathcal{L}_{\pm 1})$ and $\mathbf{g}_{2k} \in \pi_{2k}(\mathcal{L}_{\pm 1})$ are nontrivial. This establishes our main result.

Theorem 2 *Let $k \geq 1$. Then $\dim H^{2k}(\mathcal{L}_{(-1)^{(k+1)}}, \mathbb{R}) \geq 2$ and $\pi_{2k}(\mathcal{L}_{(-1)^{(k+1)}})$ contains a copy of $\mathbb{Z}$.*

Notice that $\mathcal{L}_{per}$, the set of all 1-periodic locally convex curves $\tilde{\gamma} : \mathbb{R} \to \mathbb{S}^2$, is homeomorphic to $SO(3) \times \mathcal{L}_I$: define $\Psi : \mathcal{L}_{per} \to SO(3) \times \mathcal{L}_I$ by $\Psi(\tilde{\gamma}) = (\mathfrak{F}_{\tilde{\gamma}}(0), (\mathfrak{F}_{\tilde{\gamma}}(0))^{-1}\tilde{\gamma}|_{[0,1]})$. We usually prefer to work in $\mathcal{L}_I$ but sometimes move to $\mathcal{L}_{per}$.

In Section 2, we give a brief sketch of Little's Theorem and present the concept of convex curves. Section 3 is dedicated to Theorem 1. In Section 4, we prove the basic facts about the set of flowers. Section 5 contains the construction of the map $\mathbf{g}_2 : \mathbb{S}^2 \to \mathcal{L}_{+1}$ and the proof of Theorem 2 for $k = 1$. The construction of the maps $\mathbf{h} : X \times [0,1] \to \mathbb{S}^3$ and $\tilde{\mathbf{h}}_0 : X \times \{0\} \to \mathcal{L}$ are presented in Section 6. Finally, in Section 7, we construct the maps $\mathbf{g}_{2k} : \mathbb{S}^{2k} \to \mathcal{L}_{(-1)^{(k+1)}}$ and prove Theorem 2 for $k > 1$. Section 8 contains a few final remarks.

In the second paper of this series ([10]) we prove that the connected components of $\mathcal{L}_I$ are simply connected and compute the groups $H^2(\mathcal{L}_{\pm 1}; \mathbb{Z})$.

This work was motivated by an attempt to extend to ordinary differential equations of order 3 some of our results with Dan Burghelea and Carlos Tomei ([2], [3]). Consider the differential equation of order 3:

$$u'''(t) + h_1(t)u'(t) + h_0(t)u(t) = 0, \quad t \in [0,1];$$

the set of pairs of potentials (h_0, h_1) for which the equation admits 3 linearly independent periodic solutions is homotopically equivalent to $\mathcal{L}_I$ ([12]). The motivation of B. Shapiro and M. Shapiro for studying these spaces is similar.

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2 Convex curves and Little's Theorem

In this section we give a brief review of Little's argument ([7]).

Given an interval I, a smooth immersion $\gamma : I \to \mathbb{S}^2$ and $t \in I$, let $\mathbf{n}(t)$ be the unit normal vector $\mathfrak{F}_\gamma(t)e_3 = \gamma(t) \times \gamma'(t)/|\gamma'(t)|$. Given $t_0 \in I$, the function

$\eta_{t_0}(t) = \langle \gamma(t), \mathbf{n}(t_0) \rangle$ satisfies $\eta_{t_0}(t_0) = \eta'_{t_0}(t_0) = 0$. The curve γ is locally convex near t_0 if and only if $\eta''_{t_0}(t_0) > 0$. A locally convex curve is *convex* if $\eta_{t_0}(t) > 0$ for all $t \neq t_0$. In other words, a convex curve is contained in one of the half spaces defined by the plane orthogonal to $\mathbf{n}(t_0)$.

Part of Little's Theorem is that the set $\mathcal{L}_{-1,c}$ of simple locally compact curves is a connected component of $\mathcal{L}_I$: Little proves that simple closed locally convex curves are convex (see also [15]). We shall often use this fact.

The other part of Little's Theorem is that, once convex curves have been removed, the sets $\mathcal{L}_{+1}$ and $\mathcal{L}_{-1,n}$ are path connected. The fundamental construction here is that if the curve γ has a loop, we can add a pair of loops as in Figure 3: from (a) to (b), the loop moves one full turn along a geodesic and from (b) to (c) the large loops are shrunk. By repeating this procedure, we may add a large number of loops which can then be spread along the curve. The curve can then be deformed and, thanks to the loops, remain locally convex. This part will be explained in greater detail in the next section.

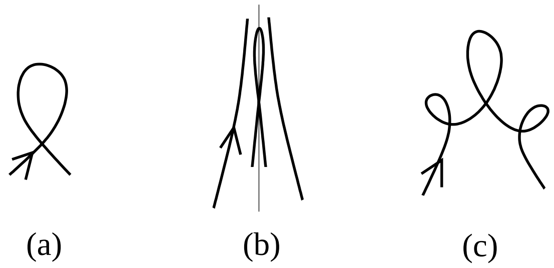

Figure 3: Using a loop to produce two more loops.

3 Proof of Theorem 1

First notice that in $\mathcal{I}_{\pm 1}$ it is easy to introduce a pair of loops at any point of the curve: the process is illustrated in Figure 4; in the final step one of the loops becomes big, goes around the sphere and shrinks again.

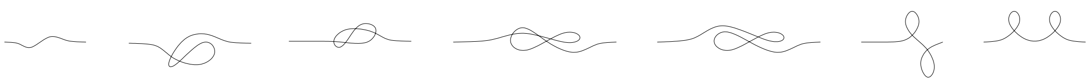

Figure 4: How to add two small loops to a curve in $\mathcal{I}_{\pm 1}$.

A function $f : K \to \mathcal{I}_{\pm 1}$ can be thought of as a family of curves. We can uniformly perform the above construction several times along all curves of the family. Given a curve γ_0, we construct a family of curves ending in a curve γ_1 with many positively oriented loops as in Figure 5. If the number of loops is sufficiently large and the loops are tight enough, the curve γ_1 will be locally convex. We have therefore constructed a homotopy $H : [0,1] \times K \to \mathcal{I}_{\pm 1}$ with $H(0,k) = f(k)$ and $H(1,k) \in \mathcal{L}_{\pm 1}$ for all $k \in K$, as required.

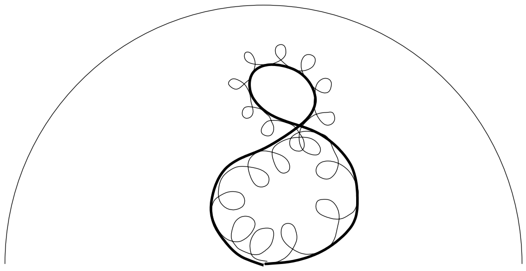

Figure 5: Curves $\gamma_0 \in \mathcal{I}_{\pm 1}$ and $\gamma_1 \in \mathcal{L}_{\pm 1}$.

We now present a more rigorous version of this argument. Let $\mathcal{C}_0$ be the circle with diameter $e_1 e_3$, parametrized by $\nu_1 \in \mathcal{L}_I$,

$$\nu_1(t) = \left(\frac{1+\cos(2\pi t)}{2}, \frac{\sqrt{2}}{2}\sin(2\pi t), \frac{1-\cos(2\pi t)}{2} \right).$$

For positive n, let $\nu_n(t) = \nu_1(nt)$ so that $\nu_1 \in \mathcal{L}_{-1,c}$ and, for $n > 1$, $\nu_n \in \mathcal{L}_{(-1)^n}$.

For $\gamma_1 \in \mathcal{I}_{\sigma_1}$, $\gamma_2 \in \mathcal{I}_{\sigma_2}$, $\sigma_i \in \{+1,-1\}$, let $\gamma_1 * \gamma_2 \in \mathcal{I}_{\sigma_1\sigma_2}$ be defined by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le 1/2, \\ \gamma_2(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Notice that if $\gamma_1, \gamma_2 \in \mathcal{L}_I$ then $\gamma_1 * \gamma_2 \in \mathcal{L}_I$. For $f : K \to \mathcal{I}_I$, let $\nu_n * f : K \to \mathcal{I}_I$ be defined by $(\nu_n * f)(p) = \nu_n * (f(p))$. The observation in Figure 4 can be translated as the following lemma, whose straightforward proof will be omitted.

Lemma 3.1 *Let K be a compact set and let $f : K \to \mathcal{I}_{\pm 1}$ a continuous function. Then f and $\nu_2 * f$ are homotopic.*

We now need a construction corresponding to adding loops along the curve, as in Figure 5. For $\gamma \in \mathcal{I}_{\pm}$ and $n > 0$, define $(F_n(\gamma))(t) = \mathfrak{F}_\gamma(t)\nu_n(t)$.

Lemma 3.2 *Let K be a compact set and let $f : K \to \mathcal{I}_{\pm 1}$ a continuous function. Then, for sufficiently large n, $F_{2n} \circ f$ is a function from K to $\mathcal{L}_{\pm 1}$.*

Proof: Let $C > 1$ be a constant such that $|(\mathfrak{F}_\gamma)'(t)| < C$ and $|(\mathfrak{F}_\gamma)''(t)| < C$ for any $\gamma = f(k)$, $k \in K$ and for any $t \in [0,1]$. Let $\epsilon > 0$ be such that if $|v_1 - \nu_1'(t)| < \epsilon$ and $|v_2 - \nu_1''(t)| < \epsilon$ then $\det(\nu_1(t), v_1, v_2) > 0$. Take $n > 20C/\epsilon$.

For $\gamma = f(k)$, write

$$\tilde{\gamma}(t) = (F_{2n}\gamma)(t) = \mathfrak{F}_\gamma(t)\nu_{2n}(t) = \mathfrak{F}_\gamma(t)\nu_1(2nt)$$

so that

$$\begin{aligned}\tilde{\gamma}'(t) &= \mathfrak{F}_\gamma'(t)\nu_1(2nt) + 2n\mathfrak{F}_\gamma(t)\nu_1'(2nt)\\ \tilde{\gamma}''(t) &= \mathfrak{F}_\gamma''(t)\nu_1(2nt) + 4n\mathfrak{F}_\gamma'(t)\nu_1'(2nt) + 4n^2\mathfrak{F}_\gamma(t)\nu_1''(2nt)\end{aligned}$$

and therefore, after a few manipulations,

$$\left|\frac{\tilde{\gamma}'(t)}{2n} - \mathfrak{F}_\gamma(t)\nu_1'(2nt)\right| < \epsilon, \qquad \left|\frac{\tilde{\gamma}''(t)}{4n^2} - \mathfrak{F}_\gamma(t)\nu_1''(2nt)\right| < \epsilon$$

or, equivalently,

$$\left|\frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}'(t)}{2n} - \nu_1'(2nt)\right| < \epsilon, \qquad \left|\frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}''(t)}{4n^2} - \nu_1''(2nt)\right| < \epsilon.$$

It follows that

$$\det\left(\nu_\theta(2nt), \frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}'(t)}{2n}, \frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}''(t)}{4n^2}\right) > 0$$

and therefore that $\det(\tilde{\gamma}(t), \tilde{\gamma}'(t), \tilde{\gamma}''(t)) > 0$, which is what we needed. ∎

Theorem 1 now follows directly from the next lemma.

Lemma 3.3 *Let K be a compact set, $f : K \to \mathcal{I}_{\pm 1}$. Then, for sufficiently large n, the image of $F_{2n} \circ f$ is contained in $\mathcal{L}_{\pm 1}$ and there exists $H : [0,1] \times K \to \mathcal{I}_{\pm 1}$ such that $H(0,\cdot) = f$ and $H(1,\cdot) = F_n \circ f$.*

Proof: We know from Lemma 3.1 that f if homotopic to $\nu_{2n} * f$. All we have to do is construct a homotopy between $F_{2n} \circ f$ and $\nu_{2n} * f$. Intuitively, this is done by pushing the loops towards $t = 0$. More precisely, if $\gamma = f(k)$, $k \in K$, let

$$H_1(s,k)(t) = \begin{cases}\nu_{2n}(t), & t \le s/2,\\ \mathfrak{F}_\gamma((2t-s)/(2-s))\nu_{2n}(t), & t \ge s/2\end{cases}$$

and

$$H_2(s,k)(t) = \begin{cases}\nu_{2n}((2t)/(2-s)), & t \le 1/2,\\ \mathfrak{F}_\gamma(2t-1)\nu_{2n}((2t)/(2-s)), & 1/2 \le t \le 1-s/2,\\ \gamma(2t-1), & t \ge 1-s/2.\end{cases}$$

Straightforward estimates of the expressions above complete the proof. ∎

This completes the proof of Theorem 1. For later use, we want a geometric understanding of what this tells us about $H^2(\mathcal{L}_{\pm 1})$.

Recall that $H_2(\mathcal{I}_{\pm 1};\mathbb{Z}) = \pi_2(\mathcal{I}_{\pm 1}) = \pi_2(\Omega\mathbb{S}^3) = \mathbb{Z}$. Since an element of $\Omega\mathbb{S}^3$ is a function from $\mathbb{S}^1$ to $\mathbb{S}^3$, a map $\alpha : \mathbb{S}^2 \to \Omega\mathbb{S}^3$ can be reinterpreted as a map $\hat{\alpha} : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$. The identification between $\pi_2(\Omega\mathbb{S}^3)$ and $\mathbb{Z}$ takes such a map α to the degree of $\hat{\alpha}$.

Similarly, let M be a closed oriented surface and consider a map $\beta : M \to \mathcal{I}_{+1}$. Let $\hat{\beta} : M \times \mathbb{S}^1 \to \mathbb{S}^3$ be defined by $\hat{\beta}(p,t) = \tilde{\mathfrak{F}}_{\beta(p)}(t)$. Define $\mathbf{x} : H_2(\mathcal{I}_{+1};\mathbb{Z}) \to \mathbb{Z}$ by $\mathbf{x}(\beta) = \deg(\hat{\beta})$: the map $\mathbf{x}$ provides the identification $H_2(\mathcal{I}_{+1};\mathbb{Z}) = \mathbb{Z}$ and is a generator of $H^2(\mathcal{I}_{+1};\mathbb{Z})$. A similar construction defines $\mathbf{x} \in H^2(\mathcal{I}_{-1};\mathbb{Z})$. The inclusion $\mathcal{L}_{\pm 1} \subset \mathcal{I}_{\pm 1}$ defines $\mathbf{x} \in H^2(\mathcal{L}_{\pm 1};\mathbb{Z})$.

As we shall see later, a function $f : K \to \mathcal{L}_I \subset \mathcal{I}_I$ may be homotopic to a constant in $\mathcal{I}_I$ but not in $\mathcal{L}_I$. The following proposition shows that this changes if we add loops.

Proposition 3.4 *Let K be a compact set and let $f : K \to \mathcal{L}_I \subset \mathcal{I}_I$ a continuous function. If f is homotopic to a constant in $\mathcal{I}_I$ then, for any $n > 0$, $\nu_n * f$ is homotopic to a constant in $\mathcal{L}_I$.*

Proof: Let $H : K \times [0,1] \to \mathcal{I}_I$ be a homotopy with $H(\cdot,0) = f$, $H(\cdot,1)$ constant. By Lemma 3.2, for sufficiently large n, say $n > N$, the image of $F_n \circ H$ is contained in $\mathcal{L}_I$. This implies that $F_n \circ f$ $(n > N)$ is homotopic in $\mathcal{L}_I$ to a constant. From Lemma 3.3, $\nu_n * f$ is homotopic to $F_n \circ f$ in $\mathcal{L}_I$ and therefore the proposition is proved for large n.

On the other hand, as we observed in Figure 3, one loop can be converted to three loops. The interval $[0,1/2]$ counts as a loop in $\nu_1 * f$ and therefore $\nu_1 * f$ is homotopic in $\mathcal{L}_I$ to $\nu_3 * f$. More generally, $\nu_n * f$ is homotopic to $\nu_{n+2} * f$, completing the proof. ∎

4 Flowers

Recall that a curve $\gamma \in \mathcal{L}_{(-1)^{(k+1)}}$ is a *flower* of $2k+1$ petals if there exist $0 = t_0 < t_1 < t_2 < \cdots < t_{2k} < t_{2k+1} = 1$ and $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{2k} < \theta_{2k+1} = \theta_M$ such that:

1. $\gamma(t_1) = \gamma(t_2) = \cdots = \gamma(t_{2k}) = e_1$;
2. the only self-intersections of the curve γ are of the form $\gamma(t_i) = \gamma(t_j)$;
3. the argument of $(x_{i,2}, x_{i,3})$ is θ_i, where $(0, x_{i,2}, x_{i,3}) = (-1)^i \gamma'(t_i)$.

For $i = 0, \ldots, 2k$, it follows from Section 2 that the restrictions $\gamma|_{[t_i,t_{i+1}]}$ (the petals) are convex curves. Let $\mathcal{F}_{2k} \subset \mathcal{L}_{(-1)^{(k+1)}}$ be the set of flowers with $2k+1$ petals.

Lemma 4.1 *The subset $\mathcal{F}_{2k} \subset \mathcal{L}_{(-1)^{(k+1)}}$ is closed. Also, there is an open neighborhood $\mathcal{A}_{2k}$ of $\mathcal{F}_{2k}$ and smooth function $\psi_{2k} : \mathcal{A}_{2k} \to \mathbb{R}^{2k}$ such that 0 is a regular value and $\mathcal{F}_{2k} = \psi_{2k}^{-1}(\{0\})$.*

Proof: We first prove that the sets $\mathcal{F}_{2k}$ are closed. Since the region near e_1 is taken care of by definition, all we have to check is that no self-tangencies within one petal or beween different petals exist in limit cases of flowers. Within each petal, a self tangency contradicts the convexity of the petal. The convexity of petals also implies that the image under γ of the interval (t_i, t_{i+1}) is contained in the open region defined by $\langle v, \mathbf{n}(t_i)\rangle > 0$, $\langle v, \mathbf{n}(t_{i+1})\rangle > 0$. Notice that these regions are disjoint and removed from each other except at the points $\pm e_1$. Thus, one petal can not touch another petal and therefore $\mathcal{F}_{2k}$ is closed.

Near a flower γ_0, curves γ will intersect the large circle through e_1 and e_2 transversally (see Figure 6). Let $\eta_0(t) = \langle \gamma(t), e_3\rangle$: we have $2k$ solutions $\hat{t}_i \approx t_i$ to $\eta_0(\hat{t}) = 0$ (where $\gamma_0(t_i) = e_1$). Define

$$\psi_{2k}(\gamma) = \left(\langle \gamma(\hat{t}_1), e_2\rangle, \langle \gamma(\hat{t}_2), e_2\rangle, \ldots, \langle \gamma(\hat{t}_{2k}), e_2\rangle\right).$$

Clearly, $\psi_{2k}(\gamma) = 0$ if and only if γ is a flower. The regularity of the value 0 follows from the fact that the curves are transversal to the horizontal plane at $\hat{t}_i$, completing the proof. ∎

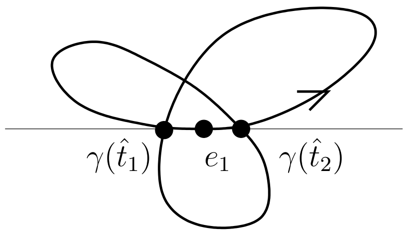

Figure 6: A curve γ near a flower with 3 petals.

The sets $\mathcal{F}_{2k}$ are contractible: this follows from the fact that the space of possible petals (convex curves) is contractible ([15]) for every choice of $t_1, \ldots, t_{2k}$ and $\theta_1, \ldots, \theta_{2k}$. We shall not use this fact in this paper; it will be proved in [10].

Counting intersections with $\mathcal{F}_{2k}$ defines an element $\mathbf{f}_{2k}$ in $H^{2k}(\mathcal{L}_{(-1)^{(k+1)}}; \mathbb{Z})$. Since the sets $\mathcal{F}_{2k}$ are disjoint and the normal bundle is trivial we have $\mathbf{f}_{2k}\mathbf{f}_{2k'} = 0$

both for $k \neq k'$ and $k = k'$. Also, the degree of $\hat{\beta} : M \times \mathbb{S}^1 \to \mathbb{S}^3$ (as in the definition of $\mathbf{x}$) can be computed at an element $z \in \mathbb{S}^3$ with

$$\Pi(z) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

since no flower ever passes through the point $-e_1$, we have $\mathbf{x}\mathbf{f}_{2k} = 0$. We still have to prove that $\mathbf{f}_{2k} \neq 0$.

5 Construction of $\mathbf{g}_2 : \mathbb{S}^2 \to \mathcal{L}_{+1}$

It is probably good to begin by recalling Little's proof that ν_2 (a circle drawn twice) and ν_4 (a circle drawn four times) are in the same connected component of $\mathcal{L}_I$. Figure 7 below illustrates this. Initially perturb your curve in order to have three self-intersection points forming approximately an equilateral triangle. Pull out the "petals" to obtain a flower with three petals, as in the third figure. Pull the petals even further so that you have a curvilinear triangle with loops at the three vertices. The passage from the fourth to the fifth figure is the only one where it is important to recall that we are in a sphere, not in the plane: one way to think of this is that the triangle became large and the bulk of the sphere passed through the triangle. Now it is merely a matter of bringing the three loops together and making the curve round again.

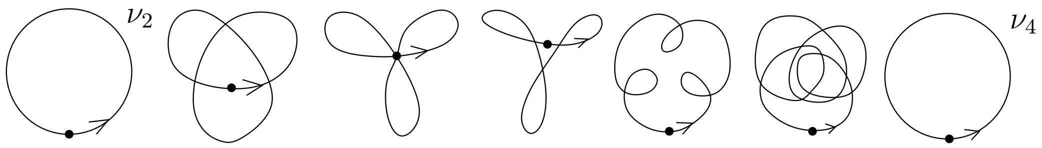

Figure 7: A path from ν_2 to ν_4.

In the construction of this path there was one important arbitrary choice: the position of the base point, or, equivalently, the orientation of the triangle. From either point of view, the construction can be turned producing a continuous family indexed by $\mathbb{S}^1$ of such paths. Since the endpoints of all paths coincide, this is equivalent to constructing a map $\mathbf{g}_{+,2} : \mathbb{S}^2 \to \mathcal{L}_{+1}$, with the two poles taken to ν_2 and ν_4, meridians (from one pole to the other) corresponding to paths like the one in Figure 7 and parallel circles being taken to paths obtained by rotating the curve an in Figure 8.

The whole construction is illustrated in Figure 9. The leftmost and rightmost columns are adjacent, the north pole ν_4 is at the top and the south pole ν_2 is

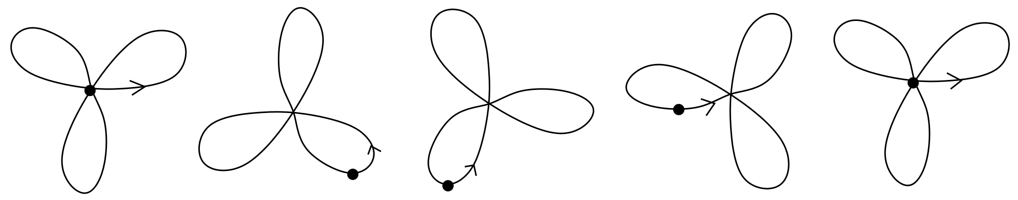

Figure 8: The image of a circle under $\mathbf{g}_{+,2}$.

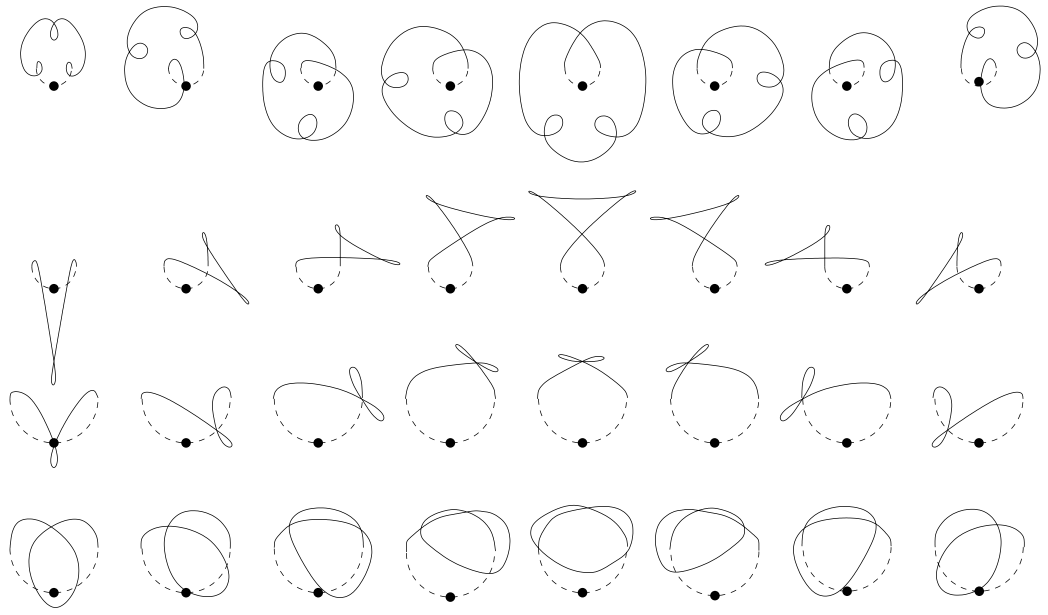

Figure 9: The function $\mathbf{g}_{+,2} : \mathbb{S}^2 \to \mathcal{L}_{+1}$.

at the bottom (as in most world maps). In the transition between the first and second rows most of the curve passed around the back of the sphere.

As an alternative to this figure, we provide a formula for $\mathbf{g}_{+,2}$. Let $\alpha : [0,1] \times [0,1] \to SO(3)$ be defined by

$$\alpha(s,t) = \begin{pmatrix} \sin \pi s \cos 2\pi t & -\sin 2\pi t & -\cos \pi s \cos 2\pi t \\ \sin \pi s \sin 2\pi t & \cos 2\pi t & -\cos \pi s \sin 2\pi t \\ \cos \pi s & 0 & \sin \pi s \end{pmatrix}$$

and define

$$\gamma_s(t) = \frac{\sqrt{2}}{2}\, \alpha(s,t) \begin{pmatrix} 1 \\ \cos 6\pi t \\ \sin 6\pi t \end{pmatrix}.$$

The curve γ_0 is a circle drawn 4 times and the curve γ_1 is a circle drawn 2 times. A tedious but straightforward computation verifies that the curves γ_s are closed and locally convex and therefore belong to $\mathcal{L}_{per}$.

Let $\Gamma(s,t) = \mathfrak{F}_{\gamma_s}(t)$: it easy to verify that

$$\Gamma(s, t+(1/3)) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Gamma(s,t)$$

for all s and t. Finally, let $\mathbf{g}_{+,2} : [0,1] \times [0,1] \to \mathcal{L}_{+1}$ be defined by

$$\mathbf{g}_{+,2}(s_1, s_2)(t) = (\Gamma(s_2, s_1/3))^{-1}\Gamma(s_2, t + (s_1/3))e_1.$$

If $s_2 = 0$ or 1, the value of s_1 is irrelevant for the value of $\mathbf{g}_{+,2}$. Also, $\mathbf{g}_{+,2}(0, s_2) = \mathbf{g}_{+,2}(1, s_2)$ for all s_2. Performing these identifications, the domain of $\mathbf{g}_{+,2}$ becomes the sphere $\mathbb{S}^2$, as required.

It follows easily either from Figure 9 or from the formulas that $\mathbf{x}(\mathbf{g}_{+,2}) = 1$, i.e., that the degree of $\hat{\mathbf{g}}_{+,2} : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$, $\hat{\mathbf{g}}_{+,2}(p,t) = \tilde{\mathfrak{F}}_{\mathbf{g}_{+,2}(p)}(t)$, equals 1. This can be seen, for instance, by computing preimages of some $z \in \mathbb{S}^3$. Thus, $\mathbf{g}_{+,2}$ is a generator of $\pi_2(\mathcal{I}_{+1})$.

It is again clear from Figure 9 that $\mathbf{g}_{+,2}$ intersects $\mathcal{F}_2$ precisely once (the flower is in the third row, first column), and the intersection is transversal, and therefore $|\mathbf{f}_2(\mathbf{g}_{+,2})| = 1$. We were not too careful about the orientation of $\mathbf{f}_2$ in Section 4 so we decree now that $\mathbf{f}_2(\mathbf{g}_{+,2}) = 1$. This can likewise be checked for the formula by a long and tedious computation which we skip. Either way, $\mathbf{f}_2 \neq 0$.

Consider the function $\nu_2 * \mathbf{g}_{+,2}$. We have $\mathbf{x}(\nu_2 * \mathbf{g}_{+,2}) = 1$ (adding these loops does not change the degree) and $\mathbf{f}_2(\nu_2 * \mathbf{g}_{+,2}) = 0$ (there are no flowers in the image of $\nu_2 * \mathbf{g}_{+,2}$ since no flower starts with two loops). In particular, the maps $\mathbf{g}_{+,2}$ and $\nu_2 * \mathbf{g}_{+,2}$ are not homotopic in $\mathcal{L}_{+1}$. On the other hand, from Lemma 3.1, the maps $\mathbf{g}_{+,2}$ and $\nu_2 * \mathbf{g}_{+,2}$ are homotopic in $\mathcal{I}_{+1}$.

We will now consider the group $\pi_2(\mathcal{L}_{+1})$ but before we do so we must say a few words about base points. Recall that given two base points p_1 and p_2, the two homotopy groups $\pi_2(\mathcal{L}_{+1}; p_1)$ and $\pi_2(\mathcal{L}_{+1}; p_2)$ are identified via a homotopy class of paths from p_1 to p_2. We prove in [10] that $\mathcal{L}_{+1}$ and $\mathcal{L}_{-1,n}$ are simply connected and therefore the identification is natural; the idea here is not, however, to use these results. We shall therefore select ν_2 as a base point for $\mathcal{L}_{+1}$ and ν_3 as a base point for $\mathcal{L}_{-1}$.

Finally, consider the difference $\mathbf{g}_2 = \mathbf{g}_{+,2} - (\nu_2 * \mathbf{g}_{+,2}) : \mathbb{S}^2 \to \mathcal{L}_{+1}$. More precisely, consider $\mathbf{g}_{+,2}$ as a function from $[0,1]^2$ to $\mathcal{L}_{+1}$ with $\mathbf{g}_{+,2}(p) = \nu_2$ for $p \in \partial([0,1]^2)$. Let $\alpha : [0,1] \to \mathcal{L}_{+1}$ be a path from ν_2 to ν_4 and define $\mathbf{g}_2 : [0,1]^2 \to \mathcal{L}_{+1}$ by

$$\mathbf{g}_2(x,y) = \begin{cases} \mathbf{g}_{+,2}(2x, y), & 0 \le x \le \frac{1}{2}; \\ (\nu_2 * \mathbf{g}_{+,2})(4x - \frac{5}{2}, 2y - \frac{1}{2}), & \max(2|x - \frac{3}{4}|, |y - \frac{1}{2}|) \le \frac{1}{4}; \\ \alpha(1 - 4\max(2|x - \frac{3}{4}|, |y - \frac{1}{2}|)), & \frac{1}{4} \le \max(2|x - \frac{3}{4}|, |y - \frac{1}{2}|) \le \frac{1}{2}. \end{cases}$$

This construction is sketched in Figure 10. From Lemma 3.1, this map is homotopic to a constant in $\mathcal{I}_{+1}$. The path α can be chosen so as to avoid the set $\mathcal{F}_2$ and therefore the image of $\mathbf{g}_2$ intersects $\mathcal{F}_2$ transversally and exactly once and we have $\mathbf{f}_2(\mathbf{g}_2) = 1$. This implies that neither $\mathbf{g}_2$ nor any positive multiple of it is homotopic to a constant in $\mathcal{L}_{+1}$. We therefore have a copy of $\mathbb{Z}^2$ contained in $\pi_2(\mathcal{L}_{+1})$. This completes the proof of the following result, closely related to the case $k = 1$ of Theorem 2.

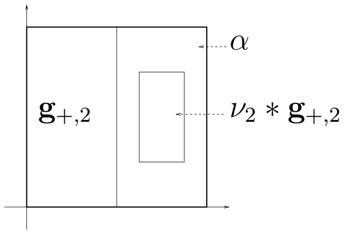

Figure 10: The construction of $\mathbf{g}_2$.

Theorem 3 *The maps $\mathbf{g}_{+,2}$ and $\mathbf{g}_2$ span a copy of $\mathbb{Z}^2$ contained in $\pi_2(\mathcal{L}_{+1})$. The elements $\mathbf{x}$ and $\mathbf{f}_2$ span a copy of $\mathbb{Z}^2$ contained in $H^2(\mathcal{L}_{+1};\mathbb{Z})$.*

6 The homotopy lifting property

Recall that $\phi : \mathcal{L} \to \mathbb{S}^3$ takes γ to $\tilde{\mathfrak{F}}_\gamma(1)$. Let $\mathbb{D}^2$ be the closed unit disk, $X_1 = \mathbb{D}^2 \times [0,1]$ and $X_2 = \mathbb{D}^2 \times \{0\} \cup \mathbb{S}^1 \times [0,1]$. We construct functions $\mathbf{h} : X_1 \to \mathbb{S}^3$ and $\tilde{\mathbf{h}}_0 : X_2 \to \mathcal{L}$ with $\phi \circ \tilde{\mathbf{h}}_0 = \mathbf{h}|_{X_2}$. We then prove that there is no continuous function $\tilde{\mathbf{h}} : X_1 \to \mathcal{L}$ with $\phi \circ \tilde{\mathbf{h}} = \mathbf{h}$, thus proving that the homotopy lifting property does not hold.

For $c \in (0, +\infty)$, let $\nu_c \in \mathcal{L}$ be defined by $\nu_c(t) = \nu_1(ct)$ so that $\phi(\nu_c) = \tilde{\mathfrak{F}}_{\nu_c}(1) = \tilde{\mathfrak{F}}_{\nu_1}(c)$,

$$\mathfrak{F}_{\nu_1}(c) = \frac{1}{2}\begin{pmatrix} 1+\cos(2\pi c) & -\sqrt{2}\sin(2\pi c) & 1-\cos(2\pi c) \\ \sqrt{2}\sin(2\pi c) & 2\cos(2\pi c) & -\sqrt{2}\sin(2\pi c) \\ 1-\cos(2\pi c) & \sqrt{2}\sin(2\pi c) & 1+\cos(2\pi c) \end{pmatrix}.$$

Define $\mathbf{h} : X_1 = \mathbb{D}^2 \times [0,1] \to \mathbb{S}^3$ by $\mathbf{h}(p,s) = \phi(\nu_{4-2s}) = \tilde{\mathfrak{F}}_{\nu_4}(1 - s/2)$; notice that $\mathbf{h}(p,0) = \mathbf{h}(p,1) = 1$ for all $p \in \mathbb{D}^2$.

Let $\pi_{\mathbb{D}^2,\mathbb{S}^2} : \mathbb{D}^2 \to \mathbb{S}^2$ be the function that wraps the sphere by taking the boundary of $\mathbb{D}$ to the north pole of $\mathbb{S}^2$, other points of $\mathbb{S}^2$ having one transversal preimage. Define $\tilde{\mathbf{h}}_0(p,0) = (\mathbf{g}_{+,2} \circ \pi_{\mathbb{D}^2,\mathbb{S}^2})(p)$ for all $p \in \mathbb{D}^2$; notice that $\tilde{\mathbf{h}}_0(p,0) = \nu_4$ for $p \in \mathbb{S}^1 = \partial\mathbb{D}^2$. Finally, for $p \in \mathbb{S}^1$, define $\tilde{\mathbf{h}}_0(p,s) = \nu_{4-2s}$ (see Figure 11).

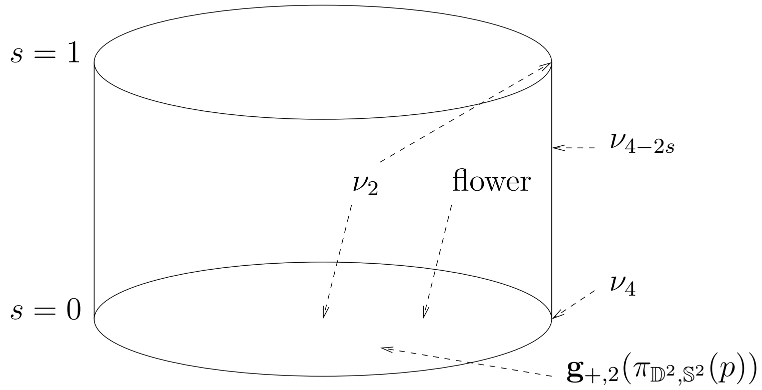

Figure 11: The function $\tilde{\mathbf{h}}_0 : X_2 = \mathbb{D}^2 \times \{0\} \cup \mathbb{S}^1 \times [0,1] \to \mathcal{L}$.

Assume by contradiction that $\tilde{\mathbf{h}} : X_1 \to \mathcal{L}$ satisfies $\phi \circ \tilde{\mathbf{h}} = \mathbf{h}$. We construct $\mathbf{h}_3 : X_1 \to \mathcal{L}_{+1}$ by completing the locally convex curves $\tilde{\mathbf{h}}(p,s)$ with a parametrized arc of $\mathcal{C}_0$. More precisely,

$$(\mathbf{h}_3(p,s))(t) = \begin{cases} (\tilde{\mathbf{h}}(p,s)) \left(\frac{t}{1-s/2}\right), & 0 \leq t \leq 1 - s/2, \\ \nu_4(t), & 1 - s/2 \leq t \leq 1; \end{cases}$$

in particular, $\mathbf{h}_3(p,s) = \nu_4$ for $p \in \mathbb{S}^1$. The construction above guarantees the continuity of $\mathbf{h}_3$.

We now consider the 2-cycle $\mathbf{h}_2 = \mathbf{h}_3|_{\partial X_1}$ and its product with $\mathbf{f}_2$. In other words, we count flowers in the image of the boundary. There is a unique flower at the image of the bottom $\mathbb{D}^2 \times \{0\}$: this intersection with $\mathcal{F}_2$ is transversal. Since $\mathbf{h}_2(p,s) = \nu_4$ for all $(p,s) \in \mathbb{S}^1 \times [0,1]$ there are no flowers on the sides. Finally, $\mathbf{h}_2(p,1)(t) = \nu_4(t)$ for all $t \geq 1/2$: curves on the top finish with two turns around $\mathcal{C}_0$ and are therefore definitely not flowers. This means that the product of $\mathbf{h}_2$ with $\mathbf{f}_2$ is not zero and therefore $\mathbf{h}_2 \neq 0 \in \pi_2(\mathcal{L}_{+1})$, contradicting the existence of $\mathbf{h}_3$.

7 Construction of $\mathbf{g}_{2k} : \mathbb{S}^{2^k} \to \mathcal{L}_{(-1)^{(k+1)}}$ and proof of Theorem 2

There is a dashed arc around the base point in each curve in Figure 9. The dashed arc remains unchanged during the entire process. We show that a minor modification of $\mathbf{g}_{+,2}$ can be constructed so that this dashed arc is a circle minus a small gap, i.e., changes are restricted to a small interval.

Take any function $f : K \to \mathcal{L}_I$, K compact. For sufficiently small $\epsilon_1 > 0$, the arcs $\gamma|_{[0,\epsilon_1]}$ and $\gamma|_{[1-\epsilon_1,1]}$ are convex for any $\gamma = f(p)$, $p \in K$. More, for sufficiently small $\epsilon_2 > 0$ the arcs $\nu_1|_{[0,\epsilon_2]}$ and $\nu_1|_{[1-\epsilon_2,1]}$ can be inserted in γ without damaging convexity. Thus a homotopy $H : [0,1] \times K \to \mathcal{L}_I$, $H(0,p) = f(p)$, changes the curves only in a small neighborhood of the base point and at the end of the homotopy we have $\gamma(t) = \nu_1(t)$ for all $\gamma = H(1,p)$, $p \in K$ and for all $t \in [0,\epsilon_2] \cup [1-\epsilon_2, 1]$.

Let R be an upper triangular matrix with positive diagonal and $\gamma \in \mathcal{L}_I$ a locally convex curve. The curve $\alpha : [0,1] \to \mathbb{R}^3$, $\alpha(t) = R^{-1}\gamma(t)$, satisfies $\det(\alpha(t), \alpha'(t), \alpha''(t)) > 0$ for all t. The curve $\gamma^R : [0,1] \to \mathbb{S}^2$, $\gamma^R(t) = \alpha(t)/|\alpha(t)|$ also satisfies $\det(\gamma^R(t), (\gamma^R)'(t), (\gamma^R)''(t)) > 0$ for all t and therefore is locally convex. The group of matrices of the form

$$R = \begin{pmatrix} a^{-1} & a^{-1}b & a^{-1}b^2/2 \\ 0 & 1 & b \\ 0 & 0 & a \end{pmatrix}, \quad a > 0,$$

takes the cone $y^2 = 2xz$ onto itself. Thus, for R as above, the small arc of the circle $\mathcal{C}_0$ around e_1 common to all curves γ, $\gamma = f(p)$, $p \in K$, is taken to another arc of $\mathcal{C}_0$ common to all curves γ^R, with another parametrization different from ν_1 but common to all curves. If a is taken to be a large positive number, the arc will be arbitrarily large, $\mathcal{C}_0$ minus a small gap; an appropriate choice of b allows us to position that gap anywhere along $\mathcal{C}_0$ away from e_1. A reparametrization allows us to assume that $\gamma(t) = \nu_1(t)$ except in a small interval $I \subset [0,1]$. Notice that this construction preserves the fact that there is only one intersection with $\mathcal{F}_2$, and this intersection is transversal.

We are now ready to construct $\mathbf{g}_{2k}$ recursively from $\mathbf{g}_2$ and $\mathbf{g}_{2k-2}$. Assume by induction that $\mathbf{g}_{2k-2}$ intersects the manifold $\mathcal{F}_{2k-2}$ transversally and exactly once so that $\mathbf{f}_{2k-2}(\mathbf{g}_{2k-2}) = 1$ and that $\nu_n * \mathbf{g}_{2k-2}$ is homotopic to a constant for any $n > 0$. We first construct $\mathbf{g}_{+,2k}$ with domain $\mathbb{S}^{2k} = \mathbb{D}^2 \times \mathbb{S}^{2k-2} \cup \mathbb{S}^1 \times \mathbb{D}^{2k-1}$.

Let $I_1 = [1/6, 2/6]$, $I_2 = [4/6, 5/6]$. Define functions $g_1 : \mathbb{D}^2 \to \mathcal{L}_{+1}$ and $g_2 : \mathbb{S}^{2k-2} \to \mathcal{L}_{(-1)^k}$ with:

(a) $g_1(p)(t) = \nu_1(t)$ for all $p \in \mathbb{D}^2$, $t \in [0,1] \smallsetminus I_1$;

(b) $g_1(p)$ is a reparametrization of ν_4 for $p \in \mathbb{S}^1 = \partial\mathbb{D}^2$;

(c) g_1 intersects $\mathcal{F}_2$ transversally and exactly once at $p_1^{\mathcal{F}} \in \int(\mathbb{D}^2)$;

(d) $g_2(p)(t) = \nu_1(t)$ for all $p \in \mathbb{S}^{2k-2}$, $t \in [0,1] \smallsetminus I_2$;

(e) g_2 intersects $\mathcal{F}_{2k-2}$ transversally and exactly once at $p_2^{\mathcal{F}} \in \mathbb{S}^{2k-2}$;

(f) $\nu_n * g_2$ is homotopic to a constant in $\mathcal{L}_I$ for $n > 0$.

The function g_1 is obtained from $\mathbf{g}_{+,2} \circ \pi_{\mathbb{D}^2,\mathbb{S}^2}$ via the above construction. Similarly, the function g_2 is obtained from $\mathbf{g}_{2k-2}$ by the same construction. For $(p_1, p_2) \in \mathbb{D}^2 \times \mathbb{S}^{2k-2}$ define

$$\mathbf{g}_{+,2k}(p_1,p_2)(t) = \begin{cases} g_1(p_1)(t), & t \in I_1, \\ g_2(p_2)(t), & t \in I_2, \\ \nu_1(t), & \text{otherwise.} \end{cases}$$

As in Figure 12, we can say that each g_j is responsible for the interval I_j. Notice that for $p_1 \in \mathbb{S}^1$, $\mathbf{g}_{+,2k}(p_1,p_2)$ is a reparametrization of $\nu_1 * (\nu_2 * g_2(p_2))$ (the reparametrization is independent of p_1). Let $g_3 : \mathbb{D}^{2k-1} \to \mathcal{L}_{(-1)^k}$ be a continuous map with $g_3(p_2) = \nu_2 * g_2(p_2)$ for all $p_2 \in \mathbb{S}^{2k-2}$. Up to the above mentioned reparametrization, for $(p_1,p_2) \in \mathbb{S}^1 \times \mathbb{D}^{2k-1}$ define $\mathbf{g}_{+,2k}(p_1,p_2) = \nu_1 * g_3(p_2)$.

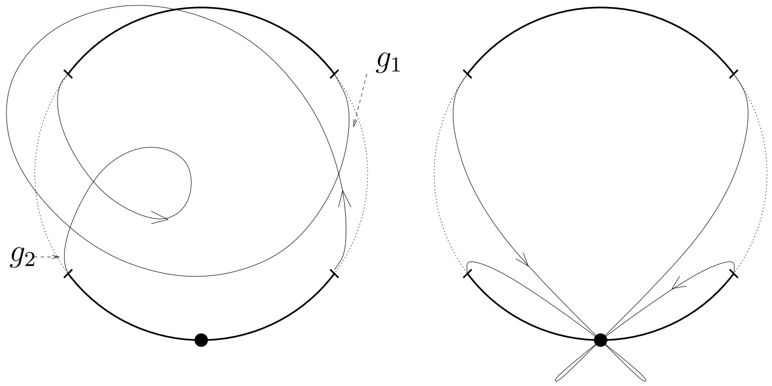

Figure 12: Two curves in the image of $\mathbf{g}_{+,2k}$; the second is a flower.

As in Figure 12, the only intersection of $\mathbf{g}_{+,2k}$ with $\mathcal{F}_{2k}$ in $\mathbb{D}^2 \times \mathbb{S}^{2k-2}$ is in $(p_1^{\mathcal{F}}, p_2^{\mathcal{F}})$; this intersection is transversal. There are no intersections of $\mathbf{g}_{+,2k}$ with $\mathcal{F}_{2k}$ in $\mathbb{S}^1 \times \mathbb{D}^{2k-1}$ for curves there are of the form $\nu_1 *$ (something).

Finally, construct $\mathbf{g}_{2k} = \mathbf{g}_{+,2k} - (\nu_2 * \mathbf{g}_{+,2k})$, as for $\mathbf{g}_2$ in Section 5. By Lemma 3.1 and Proposition 3.4, $\mathbf{g}_{2k}$ is homotopic to a constant in $\mathcal{I}_I$ or, equivalently, $\nu_2 * \mathbf{g}_{2k}$ is homotopic to a constant in $\mathcal{L}_I$. There are no flowers in the image of $\nu_2 * \mathbf{g}_{+,2k}$ and therefore $\mathbf{f}_{2k}(\mathbf{g}_{2k}) = 1$. This completes the inductive construction of $\mathbf{g}_{2k}$, proves that $\mathbf{f}_{2k} \neq 0$ and completes the proof of Theorem 2. We sum up some of our other conclusions as another theorem.

Theorem 4 *Let $k \geq 1$. Consider the inclusion $i : \mathcal{L}_{(-1)^{(k+1)}} \to \mathcal{I}_{(-1)^{(k+1)}}$ and the induced map $\pi_{2k}(i) : \pi_{2k}(\mathcal{L}_{(-1)^{(k+1)}}) \to \pi_{2k}(\mathcal{I}_{(-1)^{(k+1)}})$. Then the map $\mathbf{g}_{2k}$ constructed above spans a copy of $\mathbb{Z}$ in $\ker(\pi_{2k}(i))$.*

8 Final remarks

In the second paper of this series ([10]) we show that connected components of $\mathcal{L}_I$ are simply connected. We also show that the inclusion $\mathcal{L}_{-1,n} \subset \mathcal{I}_{-1}$ induces an isomorphism between $\pi_2(\mathcal{L}_{-1,n})$ and $\pi_2(\mathcal{I}_{-1}) = \mathbb{Z}$ and that $\mathbf{g}_2$ and $\tilde{\mathbf{g}}_2$ (as in Section 5) actually generate $\pi_2(\mathcal{L}_{+1}) = \mathbb{Z}^2$. This implies that $H^2(\mathcal{L}_{+1}; \mathbb{Z}) = \mathbb{Z}^2$ and $H^2(\mathcal{L}_{-1}; \mathbb{Z}) = \mathbb{Z}$. In [9] we hope to prove that the classes $\mathbf{x}^n$ and $\mathbf{f}_{2n}$ are generators of $H^*(\mathcal{L}_{\pm 1})$ and that $\mathcal{L}_{+1}$ and $\mathcal{L}_{-1,n}$ have the homotopy type of $\Omega\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \cdots$ and $\Omega\mathbb{S}^3 \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \mathbb{S}^{12} \vee \cdots$, respectively.

Little's Theorem that convex curves form a separate connected component can be rephrased as saying that $\mathcal{F}_0$, the set of flowers with 1 petal, obtains a new element in $H^0(\mathcal{L}_I)$. From this point of view that result is the case $k = 0$ of Theorem 2.

The sets $\mathcal{L}_{+1}$ and $\mathcal{L}_{-1}$ can naturally be considered as two instances of a big family of spaces $\mathcal{L}_z = \phi^{-1}(\{z\})$, $z \in \mathbb{S}^3$ where $\phi : \mathcal{L} \to \mathbb{S}^3$ takes γ to $\tilde{\mathfrak{F}}_\gamma(1)$. As we saw in Section 6, this map does not satisfy the homotopy lifting property. These results also imply that Gromov's h-principle ([4], [5]) fails for $\phi : \mathcal{L} \to \mathbb{S}^3$. In [11], on the other hand, we show that every space $\mathcal{L}_z$ is homeomorphic to either $\mathcal{L}_{+1}$, $\mathcal{L}_{-1}$ or $\mathcal{I}_{+1}$. Since $\mathcal{I}_{+1}$ is well understood, this leaves out only the two spaces studied in this paper.

Finally, similar questions can be asked about curves in $\mathbb{S}^n$, $n > 2$ (γ is locally convex if $\det(\gamma(t), \ldots, \gamma^{(n)}(t)) > 0$); in [11] we show a few results about these spaces.

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