The Accumulated Distribution of Quadratic Forms on the Sphere

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Abstract: Let A be a real, symmetric matrix and consider the quadratic form $q(v) = \langle Av, v \rangle$ restricted to the unit sphere. Let $Q(t)$ be the area in the unit sphere of the set of vectors v with $q(v) \leq t$. In this paper, we study the smoothness properties of Q: it fails to be analytic at the eigenvalues of A, and we give a detailed description of its behavior at such points.

Consider the following experiment. Let $A = \text{diag}(0, \lambda, 1)$, for $0 < \lambda < 1$, $q : \mathbf{S}^2 \subset \mathbf{R}^3 \to \mathbf{R}$ the quadratic form $q(v) = v^T Av$. Compute the values of q for a random collection of uniformly distributed vectors over the sphere and graph its accumulated distribution $Q : \mathbf{R} \to \mathbf{R}$, given below for a sample of 2000 points.

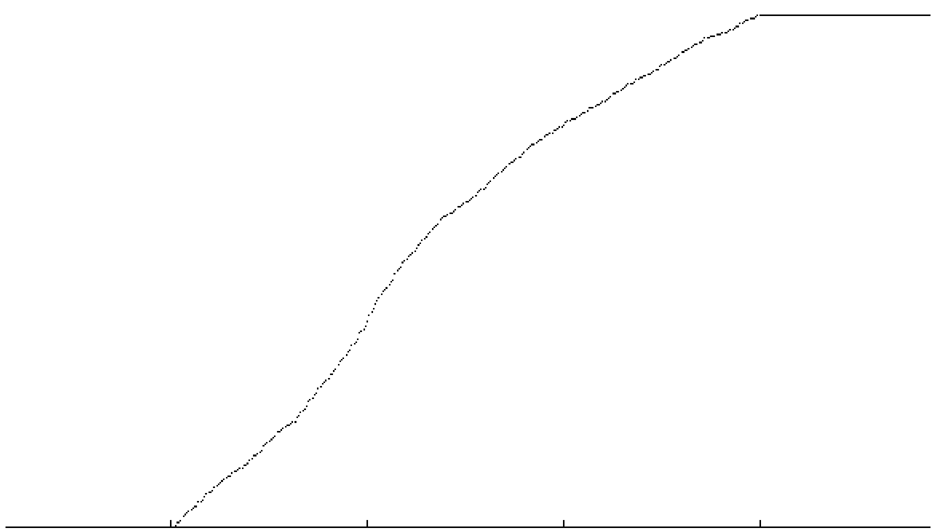

Fig. 1

As the reader can (correctly) guess by looking at this graph, Q fails to be smooth exactly at the spectrum of A: for the extreme eigenvalues, left and right derivatives are different, while for the central eigenvalue, both side derivatives are equal to infinity. In this paper we state and prove the generalization of this result to arbitrary dimensions. Let A be a real symmetric matrix of size N with spectrum $\lambda_1 \leq \ldots \leq \lambda_N$, q the associated quadratic form restricted to the unit sphere in $\mathbf{R}^N$ and Q the accumulated distribution of q,

$$Q(t) = \frac{1}{\mathrm{vol}(\mathbf{S}^{N-1})} \int_{D_t} dV,$$

where dV is the standard volume form on the unit sphere $\mathbf{S}^{N-1}$ and $D_t = \{v \in \mathbf{S}^{N-1} \mid q(v) < t\}$. We shall use Iverson's notation and write

$$[\mathcal{P}] = \begin{cases} 1 & \text{, if } \mathcal{P} \text{ holds,} \\ 0 & \text{, otherwise,} \end{cases}$$

where $\mathcal{P}$ is any condition. Set V_n to be the surface of the unit sphere $\mathbf{S}^{n-1}$ in $\mathbf{R}^n$.

Theorem: *The accumulated distribution Q is real analytic outside the spectrum of A. Let λ be an eigenvalue of multiplicity $\ell = N - n - m$ such that the spectrum of A contains m eigenvalues strictly smaller than λ and n eigenvalues strictly larger than λ. Then Q can be written in a neighborhood of λ in a unique way as*

$$Q(t) = g_0(t) + |t-\lambda|(t-\lambda)^{\frac{n+m}{2}-1} g_1(t) + (t-\lambda)^{\frac{n+m}{2}} \log(|t-\lambda|) g_2(t)$$

if $n+m$ is even or as

$$Q(t) = g_0(t) + [t > \lambda]|t-\lambda|^{\frac{n+m}{2}} g_3(t) + [t < \lambda]|t-\lambda|^{\frac{n+m}{2}} g_4(t)$$

if $n+m$ is odd, where all functions g_i are real analytic near λ. Set $p_\lambda = \prod_{\lambda' \neq \lambda}(\lambda' - \lambda)$. For $n+m$ even,

$$g_1(\lambda) = [n \text{ is even}](-1)^{\frac{m}{2}} \frac{1}{2} \frac{V_\ell V_{n+m}}{(n+m)\sqrt{|p_\lambda|}},$$

$$g_2(\lambda) = [n \text{ is odd}](-1)^{\frac{m+1}{2}} \frac{1}{\pi} \frac{V_\ell V_{n+m}}{(n+m)\sqrt{|p_\lambda|}},$$

and for $n+m$ odd,

$$g_3(\lambda) = [n \text{ is odd}](-1)^{\frac{m}{2}} \frac{V_\ell V_{n+m}}{(n+m)\sqrt{|p_\lambda|}},$$

$$g_4(\lambda) = [n \text{ is even}](-1)^{\frac{n}{2}+1} \frac{V_\ell V_{n+m}}{(n+m)\sqrt{|p_\lambda|}}.$$

Notice that exactly one of $g_1(\lambda)$ and $g_2(\lambda)$ is zero when $n+m$ is even and that exactly one of $g_3(\lambda)$ and $g_4(\lambda)$ is zero when $n+m$ is odd. This implies that $Q(t)$ is of class $C^{\lceil \frac{n+m}{2} \rceil - 1}$ but not of class $C^{\lceil \frac{n+m}{2} \rceil}$ at λ.

Uniqueness of the analytic functions g_i is trivial given existence. The theorem follows from an argument full of calculations, which we split in steps.

Step 0: Analyticity of Q outside the spectrum of A.

For convenience, set $D_{t_0,t_1} = D_{t_1} - D_{t_0}$. Let $t_0 < t_1$ be two regular values of q with no eigenvalues of A in between. Let $L_t = \{v \in \mathbf{S}^{N-1} \mid q(v) = t\}$. Notice that L_t for $t \in [t_0, t_1]$ is an analytic compact manifold. The gradient of q, after normalization, gives rise to a flow defining an analytic diffeomorphism ϕ_0 between $L_{t_0} \times [t_0, t_1)$ and D_{t_0,t_1}: it takes the pair (v_0, t) to the point v with $q(v) = t$, which can be reached from v_0 by following the normalized gradient (a good reference for this kind of construction is [M]). We must then have

$$Q(t) = Q(t_0) + \int_{t_0}^{t} \int_{L_{t_0}} |\det \phi_0'(v,s)| \, dv \, ds,$$

by the change of variables induced by ϕ_0: notice that $|\det \phi_0'|$ is analytic, since it is never zero.

Step 1: Behavior of Q at extreme eigenvalues of A.

Without loss, we consider Q close to the smallest eigenvalue $\lambda = \lambda_1$ with multiplicity ℓ. We split vectors in $\mathbf{R}^N$ as $v = (z, x) = (z_1, \ldots, z_\ell, x_1, \ldots, x_n)$ and then the quadratic form becomes

$$q(v) = \lambda + \sum_{1 \le i \le n} (\lambda_{\ell+i} - \lambda) x_i^2,$$

since v is of unit length. Notice that, as expected, $q(v) \ge \lambda$. For t slightly larger than λ, we want to compute $Q(t) = \int_{D_t} dV$. In order to perform a change of variables, we parametrize D_t by the diffeomorphism $\phi_1 : \mathbf{S}^{\ell-1} \times E_t \to D_t$ with $\phi_1(\gamma, x) = \left(\sqrt{1 - |x|^2}\, \gamma, x\right)$, where

$$E_t = \left\{ x \in \mathbf{R}^n \;\Big|\; \sum_i (\lambda_{\ell+i} - \lambda) x_i^2 < t - \lambda \right\}.$$

We now compute how the volume element changes under ϕ_1. We construct a convenient orthogonal basis of tangent vectors at the point (γ, x) for $x \neq 0$. First, take $\ell - 1$ vectors of the form $(\dot\gamma, 0)$, where $\dot\gamma$ is necessarily orthogonal to γ. Next take $n - 1$ vectors of the form $(0, \dot x)$, where the $\dot x$ are taken to be orthogonal to $x \in \mathbf{R}^n$. The last vector is $(0, x)$. By taking derivatives, we see that this basis is taken to the orthogonal basis of vectors of the form $(\sqrt{1 - |x|^2}\, \dot\gamma, 0)$, $(0, \dot x)$ and $\left((-|x|^2/\sqrt{1 - |x|^2})\gamma, x\right)$. It then immediately follows that the absolute value of the determinant of the Jacobian matrix of ϕ_1 at the point (γ, x) is $(1 - |x|^2)^{\frac{\ell-2}{2}}$. Therefore

$$Q(t) = V_\ell \int_{E_t} (1 - |x|^2)^{\frac{\ell-2}{2}} dV.$$

Now parametrize the ellipsoid E_t by the ball $B_{\sqrt{t-\lambda}} = \{w \in \mathbf{R}^n \mid \sum_i w_i^2 \leq (t-\lambda)^2\}$ by the natural linear change of coordinates with determinant given by $1/\sqrt{|p_\lambda|}$, where p_λ is as defined in the statement of the theorem. We then have

$$\begin{aligned} Q(t) &= \frac{V_\ell}{\sqrt{|p_\lambda|}} \int_{B_{\sqrt{t-\lambda}}} \left(1 - \sum_i \frac{w_i^2}{\lambda_{\ell+i}-\lambda}\right)^{\frac{\ell-2}{2}} dV \\ &= \frac{V_\ell}{\sqrt{|p_\lambda|}} \int_0^{\sqrt{t-\lambda}} r^{n-1} \int_{\mathbf{S}^{n-1}} \left(1 - \sum_i \frac{(r\theta_i)^2}{\lambda_{\ell+i}-\lambda}\right)^{\frac{\ell-2}{2}} d\theta\, dr \\ &= \frac{V_\ell V_n}{\sqrt{|p_\lambda|}} \int_0^{\sqrt{t-\lambda}} r^{n-1} h_0(r) dr, \end{aligned}$$

where $\theta = (\theta_1, \ldots, \theta_n) \in \mathbf{S}^{n-1}$ and

$$h_0(r) = \frac{1}{V_n} \int_{\mathbf{S}^{n-1}} \left(1 - \sum_i \frac{(r\theta_i)^2}{\lambda_{\ell+i}-\lambda}\right)^{\frac{\ell-2}{2}} d\theta.$$

Notice that $h_0(r)$ is the average over a sphere of radius r around the origin of the analytic function of w given by $\left(1 - \sum_i \frac{w_i^2}{\lambda_{\ell+i}-\lambda}\right)^{\frac{\ell-2}{2}}$: this last function has value one and derivative zero at the origin. Hence h_0 when extended to the negative numbers as an even function is a real analytic function with $h_0(0) = 1$ and $h_0'(0) = 0$. We must then have

$$Q(t) = [t > \lambda] \frac{V_\ell V_n}{n\sqrt{|p_\lambda|}} (t-\lambda)^{n/2} h_1(t)$$

where h_1 is an analytic function with $h_1(\lambda) = 1$. This completes step 1.

We are left with the case when λ is not an extreme eigenvalue of A. Let λ be an eigenvalue of multiplicity ℓ such that there are m strictly smaller eigenvalues and n strictly larger eigenvalues, always counted with multiplicities. Write $v = (y, z, x) = (y_1, \ldots, y_m, z_1, \ldots, z_\ell, x_1, \ldots, x_n)$. Again, since v is a unit vector, we have

$$q(v) = \lambda + q_x(v) - q_y(v),$$

where

$$q_x(v) = \sum_{1 \leq i \leq n} (\lambda_{m+\ell+i} - \lambda) x_i^2, \qquad q_y(v) = \sum_{1 \leq j \leq m} (\lambda - \lambda_j) y_j^2.$$

Each term inside the summations is greater or equal to zero.

We now illustrate our approach to the study of $Q(t)$ for t near λ in a simple low-dimensional example. Let $A = \operatorname{diag}(0, 1, 3)$ and consider the level sets $q(v) = 0.9$, $q(v) = 1$ and $q(v) = 1.1$ as shown in Figure 2. Notice that, for t slightly greater than 1, $Q(t) =$

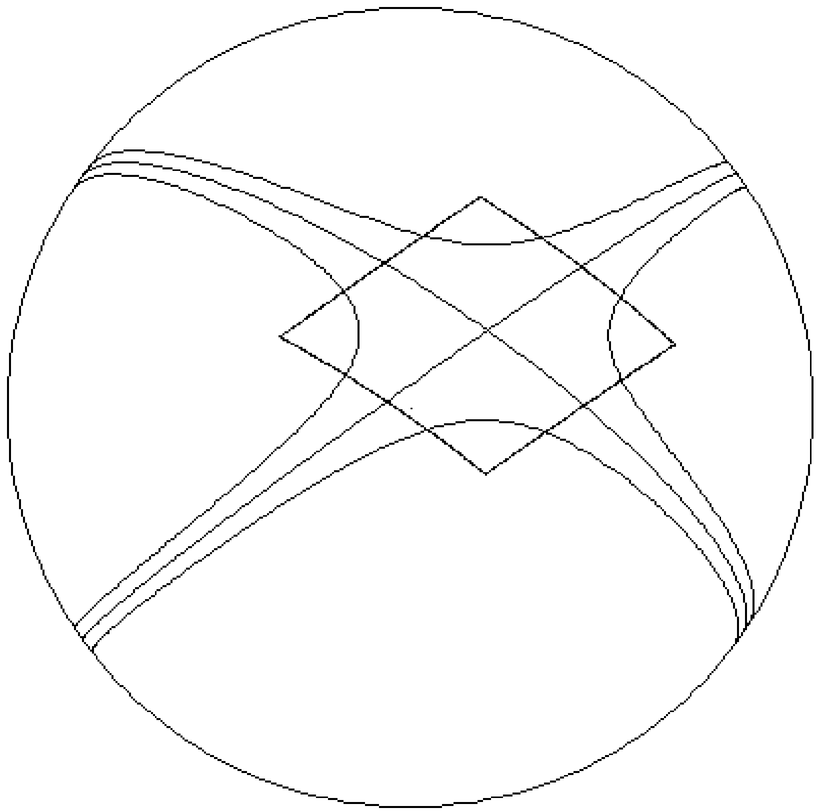

Fig. 2

$Q(1)+\text{Area}(D_{1,t})$ and the region $D_{1,t}$ is bounded by the level sets of 1 and t. Similarly, for t slightly smaller than 1, $Q(t) = Q(1) - \text{Area}(D_{t,1})$ and the region $D_{t,1}$ is again bounded by the level sets of 1 and t. The areas to be computed break into parts as shown in Figure 2. If the portion of the boundary of H intersecting relevant level sets is analytic we expect the area outside H to vary analytically in t, as is to be proved in a moment for the general case. We must therefore compute the area inside H, inducing us to choose H so that computations will be feasible.

It will be convenient to define a function $\eta : \mathbf{S}^N \to \mathbf{R}$ such that $\eta(v) = \sqrt{q_x(v)} + \sqrt{q_y(v)}$, which is clearly analytic whenever the vectors x and y are both non-zero. Let $H = \{v = (y,z,x) \in \mathbf{S}^N | \eta(v) \leq R\}$, where R is a sufficiently small fixed number. More precisely, we ask R to be small enough so that, say, $|z|^2 = \sum_{1 \leq k \leq \ell} z_k^2 > \frac{1}{2}$ for all $(y,z,x) \in H$; we shall find it convenient to demand some further conditions on R later on. We now write $Q_0(t) = \int_{D_t \cap H} dV$, $Q_1(t) = \int_{D_t - H} dV$ and $Q(t) = Q_0(t) + Q_1(t)$. We claim that for t inside a fixed open neighborhood of λ the $Q_1(t)$ is analytic so that our study reduces to $Q_0(t)$.

Step 2: Analyticity of Q_1 near λ.

The analyticity of $Q_1(t)$ can be proved in two distinct ways, one more geometric and conceptual, which we now sketch; the second one, which is more elementary and explicit, will be detailed later. The variables $t_0 < \lambda < t_1$ will be chosen close enough to λ to guarantee that our construction works well. Let L°_λ be an open subset of $L_\lambda = \{v | q(v) = \lambda\}$ consisting of regular points of q only. As in the case of t outside the spectrum of A, we can easily construct a diffeomorphism from $L^\circ_\lambda \times [t_0, t_1)$ to (a subset of) D_{t_0,t_1}, provided there are no other eigenvalues between t_0 and t_1. If we take L°_λ large and t_0 and t_1 close to λ, the image of this function contains $D_{t_0,t_1} - H$. The boundary of H is analytic and transversal to levels of q, implying, after pulling back by the diffeomorphism above, the desired result.

We now complete step 2. The idea is to parametrize $D_{t_0,t_1} - H$ in order to express $Q_1(t)$ in a manner explicit enough to infer the claim. Notice that there exists $C < 1$, depending on the spectrum of A and the choice of R only, such that if $|z| \geq C$ then $v \in H$. Define a map $\phi_2 : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \times \mathbf{B}^\ell_C \times [\lambda - C', \lambda + C'] \to \mathbf{S}^N$, taking (α, β, z, t) to a vector $v = (s_y \tilde{y}, z, s_x \tilde{x})$, where $\mathbf{B}^\ell_C = \{z \in \mathbf{R}^\ell | |z| \leq C\}$, $\tilde{x}_i = \frac{1}{\sqrt{\lambda_{m+\ell+i} - \lambda}} \alpha_i$, $\tilde{y}_j = \frac{1}{\sqrt{\lambda - \lambda_j}} \beta_j$ and the positive numbers s_x and s_y are chosen so that $v \in \mathbf{S}^N$ and $q(v) = t$:

$$s_x^2 = \frac{(1 - |z|^2) + |\tilde{y}|^2 (t - \lambda)}{|\tilde{x}|^2 + |\tilde{y}|^2}, \qquad s_y^2 = \frac{(1 - |z|^2) - |\tilde{x}|^2 (t - \lambda)}{|\tilde{x}|^2 + |\tilde{y}|^2}.$$

To ensure well-definedness and analyticity of ϕ_2 all we need is to take C' small enough to guarantee the positivity of the right hand sides of the formulas above. It is not hard to see that, under these conditions, ϕ_2 is a diffeomorphism to a subset of $\mathbf{S}^N$ containing $D_{\lambda - C', \lambda + C'} - H$; however, ϕ_2 does not parametrize this region in a way which is appropriate for computations due to the boundary arising from the removal of H. Let $\psi_0 : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \times \mathbf{S}^{\ell-1} \times [0, C] \times [\lambda - C', \lambda + C'] \to \mathbf{S}^N$ be given by $\psi_0(\alpha, \beta, \gamma, \rho, t) = \phi_2(\alpha, \beta, \rho\gamma, t)$. Notice that the function $\eta \circ \psi_0$ is analytic and strictly decreasing, in fact with strictly negative partial derivative with respect to the variable ρ. By the implicit function theorem we can solve for ρ in the equation $(\eta \circ \psi_0)(\alpha, \beta, \gamma, \rho, t) = R$, thus defining an analytic function $f_0(\alpha, \beta, \gamma, t) = \rho$. We then have, for $t \in [\lambda - C', \lambda + C']$,

$$Q_1(t) = Q_1(\lambda - C') + \\ + \int_{\lambda - C'}^{t} \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \times \mathbf{S}^{\ell-1}} \int_0^{f_0(\alpha,\beta,\gamma,s)} |\det \psi_0'(\alpha, \beta, \gamma, \rho, s)| \, d\rho \, d(\alpha, \beta, \gamma) \, ds,$$

which is obviously analytic.

Step 3: A convenient expression for Q_0 near λ.

We begin with a parametrization of H by a product of a subset H_0 of $\mathbf{R}^{n+m}$ with a sphere. Write a point of $\mathbf{R}^{n+m}$ as $(\tilde{x}, \tilde{y}) = (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_m)$; we have, of course, $|\tilde{x}|^2 = \sum_{1 \leq i \leq n} \tilde{x}_i^2$ and $|\tilde{y}|^2 = \sum_{1 \leq j \leq m} \tilde{y}_j^2$. Define $H_0 \subset \mathbf{R}^{n+m}$ by $H_0 = \left\{(\tilde{x}, \tilde{y}) \in \mathbf{R}^{n+m} \mid |\tilde{x}| + |\tilde{y}| \leq R\right\}$. Let $\phi_3 : H_0 \times \mathbf{S}^{\ell-1} \to H$ be the diffeomorphism taking $((\tilde{x}, \tilde{y}), \gamma)$ to (y, z, x) where $x_i = \frac{1}{\sqrt{\lambda_{m+\ell+i} - \lambda}} \tilde{x}_i$, $y_j = \frac{1}{\sqrt{\lambda - \lambda_j}} \tilde{y}_j$ and $z = \sqrt{1 - |x|^2 - |y|^2} \, \gamma$. The well-definedness and analyticity of ϕ_3 follow from the condition that $|z|^2 > \frac{1}{2}$ for points in H. We have that $|\det \phi_3'|$ is a positive analytic function depending on $\tilde{x}$ and $\tilde{y}$ only; it can be written as $1/\sqrt{|p_\lambda|}\ (1+(|\tilde{x}|^2+|\tilde{y}|^2)f_1(\tilde{x},\tilde{y}))$ where f_1 is an analytic (and therefore bounded) function; notice furthermore that f_1 is even. We have $Q_0(t) = Q_0(\lambda) + \left(V_n V_m V_\ell/\sqrt{|p_\lambda|}\right) F(t-\lambda)$ where

$$F(\tau) = \frac{1}{V_n V_m} \int_{\substack{(\tilde{x},\tilde{y})\in H_0 \\ 0\le|\tilde{x}|^2-|\tilde{y}|^2\le\tau}} \left(1+(|\tilde{x}|^2+|\tilde{y}|^2)f_1(\tilde{x},\tilde{y})\right) d(\tilde{x},\tilde{y}), \qquad \tau\ge 0,$$
$$F(\tau) = -\frac{1}{V_n V_m} \int_{\substack{(\tilde{x},\tilde{y})\in H_0 \\ \tau\le|\tilde{x}|^2-|\tilde{y}|^2\le 0}} \left(1+(|\tilde{x}|^2+|\tilde{y}|^2)f_1(\tilde{x},\tilde{y})\right) d(\tilde{x},\tilde{y}), \qquad \tau\le 0.$$

In order to write F more explicitly we want to parametrize H_0. Let $\psi_1 : \mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\times[-1,1]\times[0,R]\to H_0$ be defined by $\psi_1(\alpha,\beta,s,r) = ((1+s)r\alpha,(1-s)r\beta)$. Notice that the quadratic form $|\tilde{x}|^2-|\tilde{y}|^2$ when applied to $\psi_1(\alpha,\beta,s,r)$ is equal to $4sr^2$. The function ψ_1 can clearly be used as a change of coordinates with $|\det\psi'(\alpha,\beta,s,r)| = 2P_{n,m}(s)r^{n+m-1}$ where $P_{n,m}(s) = (1+s)^{n-1}(1-s)^{m-1}$: indeed, use polar coordinates to write $\tilde{x} = r_a\alpha$ and $\tilde{y} = r_b\beta$ and conclude that $d\tilde{x} = r_a^{n-1}dr_a d\alpha$ and $d\tilde{y} = r_b^{m-1}dr_b d\beta$. In our new coordinates, $f_1(\tilde{x},\tilde{y})$ should be written as $f_1((1+s)r\alpha,(1-s)r\beta)$; taking averages, set

$$f_2(s,r^2) = \frac{1}{V_n V_m}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f_1((1+s)r\alpha,(1-s)r\beta)d(\alpha,\beta),$$

which is a bounded analytic function because f_1 is even. We then have

$$F(\tau) = 2\iint_{\substack{0\le 4sr^2\le\tau \\ -1\le s\le 1, 0\le r\le R}} P_{n,m}(s)r^{n+m-1}(1+r^2 f_2(s,r^2))\,dr\,ds, \qquad \tau\ge 0,$$
$$F(\tau) = -2\iint_{\substack{\tau\le 4sr^2\le 0 \\ -1\le s\le 1, 0\le r\le R}} P_{n,m}(s)r^{n+m-1}(1+r^2 f_2(s,r^2))\,dr\,ds, \qquad \tau\le 0.$$

Split integrands and write $F(\tau) = F_0(\tau)+F_1(\tau)$ where F_0 corresponds to 1 and F_1 corresponds to $r^2 f_2(s,r^2)$:

$$\begin{aligned} F_0(\tau) = {} & 2\int_0^{\frac{\tau}{4R^2}}\int_0^R P_{n,m}(s)r^{n+m-1}\,dr\,ds\ - \\ & -2\int_{\operatorname{sgn}(\tau)}^{\frac{\tau}{4R^2}}\int_0^{\frac{1}{2}\sqrt{\frac{\tau}{s}}} P_{n,m}(s)r^{n+m-1}\,dr\,ds \end{aligned}$$

and

$$\begin{aligned} F_1(\tau) = {} & 2\int_0^{\frac{\tau}{4R^2}}\int_0^R P_{n,m}(s)r^{n+m+1}f_2(s,r^2)\,dr\,ds\ - \\ & -2\int_{\operatorname{sgn}(\tau)}^{\frac{\tau}{4R^2}}\int_0^{\frac{1}{2}\sqrt{\frac{\tau}{s}}} P_{n,m}(s)r^{n+m+1}f_2(s,r^2)\,dr\,ds, \end{aligned}$$

where $\operatorname{sgn}(\tau)$, the sign of τ, is 1 for τ positive and -1 for τ negative.

Step 4: Behavior of F_0.

Integrating with respect to r, we get

$$F_0(\tau) = \frac{2R^{n+m}}{n+m}\int_0^{\frac{\tau}{4R^2}} P_{n,m}(s)ds - \frac{1}{2^{n+m-1}(n+m)}\int_{\mathrm{sgn}(\tau)}^{\frac{\tau}{4R^2}} P_{n,m}(s)\left(\frac{\tau}{s}\right)^{\frac{n+m}{2}} ds;$$

we do not have to worry about non-integer exponents because s and τ always have the same sign. The first integral is a polynomial in τ, so we are left with computing

$$\tilde{F}_0(\tau) = \int_{\mathrm{sgn}(\tau)}^{\frac{\tau}{4R^2}} P_{n,m}(s)\left(\frac{\tau}{s}\right)^{\frac{n+m}{2}} ds.$$

Write

$$P_{n,m}(s) = (1+s)^{n-1}(1-s)^{m-1} = \sum_{1\le\iota\le n+m-1} a_{n,m,\iota}s^{\iota-1}$$

and integrate to get, for $n+m$ even,

$$\begin{aligned}
\tilde{F}_0(\tau) = &\sum_{\substack{1\le\iota\le n+m-1\\ \iota\neq\frac{n+m}{2}}} \frac{a_{n,m,\iota}}{\left(\iota-\frac{n+m}{2}\right)(4R^2)^{\iota-\frac{n+m}{2}}}\tau^{\iota} \\
&- a_{n,m,\frac{n+m}{2}}\log(4R^2)\tau^{\frac{n+m}{2}} \\
&- \left(\sum_{\substack{1\le\iota\le n+m-1\\ \iota\neq\frac{n+m}{2}}} \frac{(\mathrm{sgn}(\tau))^{\iota}a_{n,m,\iota}}{\iota-\frac{n+m}{2}}\right)|\tau|^{\frac{n+m}{2}} \\
&+ a_{n,m,\frac{n+m}{2}}\log(|\tau|)\tau^{\frac{n+m}{2}} \\
= &\sum_{\substack{1\le\iota\le n+m-1\\ \iota\neq\frac{n+m}{2}}} \frac{a_{n,m,\iota}}{\left(\iota-\frac{n+m}{2}\right)(4R^2)^{\iota-\frac{n+m}{2}}}\tau^{\iota} \\
&- \left(a_{n,m,\frac{n+m}{2}}\log(4R^2) + \sum_{\substack{1\le\iota\le n+m-1\\ \iota\neq\frac{n+m}{2},\,\iota\equiv\frac{n+m}{2} \pmod 2}} \frac{a_{n,m,\iota}}{\iota-\frac{n+m}{2}}\right)\tau^{\frac{n+m}{2}} \\
&- \left(\sum_{\substack{1\le\iota\le n+m-1\\ \iota\not\equiv\frac{n+m}{2} \pmod 2}} \frac{a_{n,m,\iota}}{\iota-\frac{n+m}{2}}\right)|\tau|\tau^{\frac{n+m}{2}-1} \\
&+ a_{n,m,\frac{n+m}{2}}\log(|\tau|)\tau^{\frac{n+m}{2}} \\
= &\text{ polynomial} + C_1|\tau|\tau^{\frac{n+m}{2}-1} + C_2\tau^{\frac{n+m}{2}}\log(|\tau|)
\end{aligned}$$

and, for $n+m$ odd,

$$\begin{aligned}
\tilde{F}_0(\tau) &= \sum_{1\le\iota\le n+m-1} \frac{a_{n,m,\iota}}{\left(\iota-\frac{n+m}{2}\right)(4R^2)^{\iota-\frac{n+m}{2}}}\tau^{\iota} \\
&\quad - [\tau>0]\left(\sum_{1\le\iota\le n+m-1}\frac{a_{n,m,\iota}}{\iota-\frac{n+m}{2}}\right)|\tau|^{\frac{n+m}{2}} \\
&\quad - [\tau<0]\left(\sum_{1\le\iota\le n+m-1}\frac{(-1)^{\iota}a_{n,m,\iota}}{\iota-\frac{n+m}{2}}\right)|\tau|^{\frac{n+m}{2}} \\
&= \text{polynomial} + C_3[\tau>0]|\tau|^{\frac{n+m}{2}} + C_4[\tau<0]|\tau|^{\frac{n+m}{2}}.
\end{aligned}$$

We are done with step 4.

Step 5: Behavior of F_1.

We can write

$$\int_0^r P_{n,m}(s)r^{n+m+1}f_2(s,r^2)\,dr = r^{n+m+2}f_3(s,r^2)$$

where f_3 is a bounded analytic function. We therefore have

$$F_1(\tau) = \int_0^{\frac{\tau}{4R^2}} R^{n+m+2}f_3(s,R^2)ds - \int_{\mathrm{sgn}(\tau)}^{\frac{\tau}{4R^2}} \left(\frac{1}{2}\sqrt{\frac{\tau}{s}}\right)^{n+m+2} f_3\left(s,\frac{\tau}{4s}\right)ds.$$

As in the previous substep, the question reduces to studying the behaviour of

$$\tilde{F}_1(\tau) = \int_{\mathrm{sgn}(\tau)}^{\frac{\tau}{4R^2}} \left(\frac{\tau}{s}\right)^{\frac{n+m}{2}+1} f_3\left(s,\frac{\tau}{4s}\right)ds.$$

More precisely, we want to prove that

$$\tilde{F}_1(\tau) = g_0^*(\tau) + |\tau|(\tau)^{\frac{n+m}{2}}g_1^*(\tau) + \tau^{\frac{n+m}{2}+1}\log(\tau)g_2^*(\tau)$$

if $n+m$ is even and that

$$\tilde{F}_1(\tau) = g_0^*(\tau) + [\tau>0]|\tau|^{\frac{n+m}{2}+1}g_3^*(\tau) + [\tau<0]|\tau|^{\frac{n+m}{2}+1}g_4^*(\tau)$$

if $n+m$ is odd, where all functions g_i^* are real analytic near 0. Let us state and prove a lemma which will take care of this problem.

Lemma: *Let a be a positive real number and $f:[-1,1]\times[0,A_0]\to\mathbf{R}$ be a real analytic function (i.e., analytic in some open neighborhood of the domain). Then there exist $A\in(0,A_0)$, $\epsilon>0$, g_0^* and,*

- *if a is an integer, there exist analytic functions g_1^* and g_2^* such that*

$$\begin{aligned} g^*(\tau) &= \int_{\mathrm{sgn}(\tau)}^{\frac{\tau}{4A}} \left(\frac{\tau}{s}\right)^a f(s, \frac{\tau}{4s}) ds \\ &= g_0^*(\tau) + |\tau|\tau^{a-1} g_1^*(\tau) + \tau^a \log(|\tau|) g_2^*(\tau). \end{aligned}$$

- *if a is not an integer, there exist analytic functions g_3^* and g_4^* such that*

$$\begin{aligned} g^*(\tau) &= \int_{\mathrm{sgn}(\tau)}^{\frac{\tau}{4A}} \left(\frac{\tau}{s}\right)^a f(s, \frac{\tau}{4s}) ds \\ &= g_0^*(\tau) + [\tau > 0]|\tau|^a g_3^*(\tau) + [\tau < 0]|\tau|^a g_4^*(\tau). \end{aligned}$$

Proof: We will use complex numbers freely. When computing real powers of complex numbers, we take the cut along the negative imaginary axis. Expand

$$f(s, w) = \sum_{\nu,\mu} b_{\nu,\mu} s^\nu w^\mu,$$

which converges absolutely and uniformly for $|s| < C$ and $|w| < 2A$, for appropriate choices of C and A:

$$\sum_{\nu,\mu} |b_{\nu,\mu}| C^\nu (2A)^\mu < +\infty.$$

Write

$$\begin{aligned} g^*(\tau) = &\int_{\mathrm{sgn}(\tau)}^{C\mathrm{sgn}(\tau)} \left(\frac{\tau}{s}\right)^a f(s, \frac{\tau}{4s}) ds \\ &+ \int_{C\mathrm{sgn}(\tau)}^{\frac{\tau}{4A}} \left(\frac{\tau}{s}\right)^a f(s, \frac{\tau}{4s}) ds. \end{aligned}$$

Call the two terms in the right hand side $g_a^*(\tau)$ and $g_b^*(\tau)$ respectively. We then have

$$g_a^*(\tau) = \tau^a \int_{\mathrm{sgn}(\tau)}^{C\mathrm{sgn}(\tau)} s^{-a} f(s, \frac{\tau}{4s}) ds,$$

where the integrand is analytic in the region of integration. When a is an integer, g_a^* is analytic and can be taken as part of g_0^*. Otherwise, write g_a^* as $[\tau > 0]|\tau|^a g_{3,a}^*(\tau) + [\tau < 0]|\tau|^a g_{a,4}^*(\tau)$, where $g_{3,a}^*$ and $g_{4,a}^*$ are analytic close to zero. Now expand f close to zero to get

$$g_b^*(\tau) = \tau^a \int_{C\mathrm{sgn}(\tau)}^{\frac{\tau}{4A}} s^{-a} \sum_{\nu,\mu} b_{\nu,\mu} s^\nu \left(\frac{\tau}{4s}\right)^\mu ds.$$

The sum and the integral commute by Fubini, yielding

$$
\begin{aligned}
g_b^*(\tau) &= \sum_{\nu,\mu} \left(\frac{b_{\nu,\mu}}{4^\mu} \tau^{a+\mu} \int_{C\mathrm{sgn}(\tau)}^{\frac{\tau}{4A}} s^{\nu-\mu-a} ds \right) \\
&= \sum_{\substack{\nu,\mu \\ \nu-\mu-a\neq -1}} \left(\frac{b_{\nu,\mu}}{4^\mu} \tau^{a+\mu} \left[\frac{s^{\nu-\mu-a+1}}{\nu-\mu-a+1} \right]_{C\mathrm{sgn}(\tau)}^{\frac{\tau}{4A}} \right) \\
&\quad + \sum_{\substack{\nu,\mu \\ \nu-\mu-a=-1}} \left(\frac{b_{\nu,\mu}}{4^\mu} \tau^{a+\mu} \left[\log s \right]_{C\mathrm{sgn}(\tau)}^{\frac{\tau}{4A}} \right).
\end{aligned}
$$

The second summand vanishes when a is not an integer; otherwise it can be written as

$$
(\log(|\tau|) + C')\tau^a \sum_{\mu} \frac{b_{\mu+a-1,\mu}}{4^\mu} \tau^\mu,
$$

where the sum is convergent for small τ, and gives rise to the term g_2^* in the statement of the lemma. The first summand is

$$
\begin{aligned}
&\sum_{\substack{\nu,\mu \\ \nu-\mu-a\neq -1}} \frac{b_{\nu,\mu}(4A)^a}{\nu-\mu-a+1} \frac{\tau}{4a}^{\nu+1} A^\mu \\
&+ \tau|\tau|^{a-1} \sum_{\substack{\nu,\mu \\ \nu-\mu-a\neq -1}} \frac{b_{\nu,\mu} C^{-a+1}}{\nu-\mu-a+1} (C\mathrm{sgn}(\tau))^\nu \left(\frac{|\tau|}{4C} \right)^\mu,
\end{aligned}
$$

where the series are convergent for sufficiently small τ, where the constants C and A have been chosen above. The first term in this expression is analytic. If a is an integer, the second term can be written in the form (analytic)$+|\tau|\tau^{a-1}$(analytic). If a is not an integer, split the second term as $[\tau > 0]|\tau|^a$(analytic) $+ [\tau < 0]|\tau|^a$(analytic).

This completes the proof of the lemma.

Step 5 now is a direct application of the lemma to the function f_3, where $A_0 = R^2$. Notice that this imposes a restriction on R which was announced above, but the function f_3 does not depend on the choice of R.

We recall the general scheme of the proof. We proved the theorem for points outside the spectrum of A and for extreme eigenvalues in steps 0 and 1. For an intermediate eigenvalue λ, we split $Q = Q_0 + Q_1$, where Q_1 is analytic (step 2). In step 3, we wrote $Q_0(t) = Q_0(\lambda) + \left(V_n V_m V_\ell / \sqrt{|p_\lambda|} \right) (F_0(t-\lambda) + F_1(t-\lambda))$. In steps 4 and 5, we studied F_0 and F_1 respectively with detail enough to obtain the claimed behavior of Q near λ. Consequently,

$$
g_i(\lambda) = -\frac{V_n V_m V_\ell}{2^{n+m-1}(n+m)\sqrt{|p_\lambda|}} C_i.
$$

Step 6: Expressions for C_i, $i = 1, 2, 3, 4$.

Notice first, using the notation introduced in step 4, that $t^{n+m-2} P_{n,m}\left(\frac{1}{t}\right) = (-1)^{m-1} P_{n,m}(t)$ and $a_{n,m,n+m-\iota} = (-1)^{m-1} a_{n,m,\iota}$. This implies that $C_2 = 0$ unless n and m are both odd, by direct evaluation of $C_2 = a_{n,m,\frac{n+m}{2}}$. Similar arguments imply that $C_1 = 0$ unless n and m are both even, $C_3 = 0$ unless n is odd and m is even and $C_4 = 0$ unless n is even and m is odd.

We now compute the value of C_2 when $n = 2\nu + 1$ and $m = 2\mu + 1$ for positive integers ν and μ. In the computations below, $\mathbf{S}^1$ stands for the unit complex circle with positive orientation:

$$\begin{aligned}
C_2 &= a_{n,m,\frac{n+m}{2}} \\
&= \frac{1}{2\pi i} \int_{\mathbf{S}^1} \frac{P_{n,m}(z)}{z^{\frac{n+m}{2}}} dz \\
&= \frac{1}{2\pi i} \int_{\mathbf{S}^1} \frac{(1+z)^{2\nu}(1-z)^{2\mu}}{z^{\nu+\mu+1}} dz \\
&= \frac{1}{2\pi 1} \int_{\mathbf{S}^1} \left(z + 2 + \frac{1}{z}\right)^{\nu} \left(z - 2 + \frac{1}{z}\right)^{\mu} \frac{dz}{z} \\
&= \frac{1}{2\pi} \int_0^{2\pi} (2 + 2\cos\theta)^{\nu} (-2 + 2\cos\theta)^{\mu} d\theta \\
&= \frac{(-1)^{\mu} 2^{2\nu+2\mu+1}}{\pi} \int_0^{\frac{\pi}{2}} \cos^{\nu}\vartheta \sin^{\mu}\vartheta d\vartheta \\
&= (-1)^{\frac{m-1}{2}} \frac{(n-1)!(m-1)!}{\left(\frac{n-1}{2}\right)! \left(\frac{m-1}{2}\right)! \left(\frac{n+m}{2} - 1\right)!}.
\end{aligned}$$

The last integral is standard and can be found, for example, in [K].

We now compute the value of C_1 when n and m are both even. Once a complex interpretation for the combinatorial expression is obtained, computations proceed as above:

$$\begin{aligned}
C_1 &= - \sum_{\substack{1 \le \iota \le n+m-1 \\ \iota \not\equiv \frac{n+m}{2} \pmod 2}} \frac{a_{n,m,\iota}}{\iota - \frac{n+m}{2}} \\
&= \frac{1}{2} \int_1^{-1} \frac{P_{n,m}(z)}{z^{\frac{n+m}{2}}} dz \\
&= (-1)^{\frac{m}{2}-1} 2^{n+m-3} \frac{\left(\frac{n}{2} - 1\right)! \left(\frac{m}{2} - 1\right)!}{\left(\frac{n+m}{2} - 1\right)!}.
\end{aligned}$$

We give a brief explanation for the integral interpretation used above. Our previous discussion shows that $a_{n,m,\frac{n+m}{2}} = 0$ so that the residue at zero is zero and there are no logs in the term by term integration. When integrating, the terms corresponding to indices appearing in the summation remain while the others cancel out.

For C_3, we have, when n is odd and m is even,

$$
\begin{aligned}
C_3 &= -\sum_{1\le\iota\le n+m-1} \frac{a_{n,m,\iota}}{\iota - \frac{n+m}{2}} \\
&= -\int_{-1}^{1} \frac{P_{n,m}(z)}{z^{\frac{n+m}{2}}} dz \\
&= (-1)^{\frac{m}{2}+1} 2^{n+2m-2} \frac{\left(\frac{m}{2}-1\right)!\left(\frac{n+m-1}{2}\right)!(n-1)!}{\left(\frac{n-1}{2}\right)!(n+m-1)!}.
\end{aligned}
$$

In the complex integral take, say, the negative imaginary axis as a cut and integrate indefinitely term by term with zero constant. The value of this integral at 1 is the required summation and its value at -1 is zero because terms cancel in pairs: term ι cancels with term $n+m-\iota$.

Finally, notice that $(-1)^{\iota} a_{n,m,\iota} = -a_{m,n,\iota}$ so that we can deduce the value of C_4 from the value of C_3:

$$
C_4 = (-1)^{\frac{n}{2}} 2^{2n+m-2} \frac{\left(\frac{n}{2}-1\right)!\left(\frac{n+m-1}{2}\right)!(m-1)!}{\left(\frac{m-1}{2}\right)!(n+m-1)!}.
$$

In order to obtain the expressions for $g_i(\lambda)$ as in the statement of the theorem, recall that the surface V_n of $\mathbf{S}^{n-1}$ is given by

$$
V_n = \begin{cases} \frac{2^n \pi^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!}{(n-1)!} & \text{, for } n \text{ odd,} \\ \frac{2\pi^{\frac{n}{2}}}{\left(\frac{n}{2}-1\right)!} & \text{, for } n \text{ even.} \end{cases}
$$

The simple algebra is left to the reader.

References:

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