Actions of $\mathbb{R}^n$ over $\mathbb{T}^n$-Fibrations

Nicolau C. Saldanha

Abstract: Let X be a smooth fibre bundle with fibre $\mathbb{T}^n$ over M^m. We want to know if there exists a smooth locally free action of $\mathbb{R}^n$ over X with orbits equal to the fibres. In this paper we give necessary and sufficient conditions for the existence of such an action. In particular, the desired action exists if X and M are orientable, $n \leq 2$ and $m \leq 2$ but may not exist if any of the three conditions is dropped.

0.Introduction

Let X be a $\mathbb{T}^n$-fibre bundle: we want to know if there exists a smooth locally free action of $\mathbb{R}^n$ over X with orbits equal to the fibres. An answer is given by the following proposition:

Proposition A: *Let X be a smooth fibre bundle with fibre $\mathbb{T}^n$ over M^m. Suppose M is a smooth compact connected manifold and that the fibers of X are coherently oriented. Then there exists a smooth locally free action of $\mathbb{R}^n$ over X with orbits equal to the fibres if and only if the following conditions hold:*

(a) It is possible to give smoothly varying affine structures to the fibres of X.

(b) The vector bundle over M with fibre $H_1(F, \mathbb{R})$ (where F is the fibre of X) is trivial.

Notice that the hypothesis of the fibres being oriented is obviously necessary but the orientability of M is not. Also, for $n = 1$ (oriented circle bundles), the desired action clearly exists: it is given by a non-zero vector field tangent to the fibres.

We investigate possible violations of conditions (a) and (b). As we shall see, condition (a) always holds for $n = 2$ but there are examples of $\mathbb{T}^7$-fibre bundles over spheres with no affine structure. Condition (b) seems to be simpler: it is related to the algebraic structure of $SL(n, \mathbb{Z}) \subseteq SL(n, \mathbb{R})$, which is very different for $n = 2$ and $n > 2$. As an application of Proposition A and the study of conditions (a) and (b) we prove:

Theorem B: *Let X be a smooth fibre bundle with fibre $\mathbb{T}^n$ over M^m. Suppose M is a smooth compact connected manifold and that the fibers of X are coherently oriented. Then, if $n \leq 2$, $m \leq 2$ and M is orientable, there exists a smooth locally free action of $\mathbb{R}^n$ over X with orbits equal to the fibres. Furthermore, if any of these three hypotheses is dropped, the conclusion does not necessarily hold.*

A torus bundle whose fibres are orbits of an action of $\mathbb{R}^n$ is a principal $\mathbb{T}^n$ bundle only in the very special case where the isotropy group of the action is constant. It is easy to see that for the $\mathbb{T}^2$ bundle over $\mathbb{S}^1$ obtained from the unit cube by identifying $(x, 0, z_2)$ with $(x, 1, z_2)$, $(x, z_1, 0)$ with $(x, z_1, 1)$ and $(0, z_1, z_2)$ with $(1, -z_2, z_1)$ there exist actions as described in the above theorems but there is no action with fixed isotropy group.

In the first section, aiming at a study of condition (b), we see some algebraic preliminaries, mostly concerning the structure of such groups as $SL(n,\mathbb{Z}) \subseteq SL(n,\mathbb{R})$ and $\widetilde{SL(n,\mathbb{Z})} \subseteq \widetilde{SL(n,\mathbb{R})}$. In the second section, having condition (a) in mind, we consider the problem of giving affine structures to the fibres of a smooth torus bundle: as we shall see, this is directly related to the (hard) problem of studying the homotopy type of $\mathrm{Diff}(\mathbb{T}^n)$, the group of smooth diffeomorphisms of the torus (as in [ABK]). Finally, in the third and fourth sections we prove Proposition A and Theorem B, respectively.

I would like to thank José Luis Arraut for suggesting the original question and Derek Hacon for helpful conversations. This research is supported by SCT and CNPq.

1. Algebraic Preliminaries

As we shall see in the last section, it is important for our problem to consider the group $\widetilde{SL(n,\mathbb{Z})}$ which is defined as follows. Let $\widetilde{SL(n,\mathbb{R})}$ be the universal covering of $SL(n,\mathbb{R})$. Using the natural inclusion $SL(n,\mathbb{Z}) \subseteq SL(n,\mathbb{R})$ we define $\widetilde{SL(n,\mathbb{Z})}$ as the preimage of $SL(n,\mathbb{Z})$ by the projection from $\widetilde{SL(n,\mathbb{R})}$ into $SL(n,\mathbb{R})$. It is well known that $\pi_1(SL(2,\mathbb{R})) = \pi_1(SO(2)) = \pi_1(\mathbb{S}^1) = \mathbb{Z}$ and that $\pi_1(SL(n,\mathbb{R})) = \pi_1(SO(n)) = \pi_1(O) = \mathbb{Z}/(2)$ for $n > 2$. We have therefore the following exact sequences of groups:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{SL(2,\mathbb{Z})} \longrightarrow SL(2,\mathbb{Z}) \longrightarrow 1$$

and, for $n > 2$,

$$1 \longrightarrow \mathbb{Z}/(2) \longrightarrow \widetilde{SL(n,\mathbb{Z})} \longrightarrow SL(n,\mathbb{Z}) \longrightarrow 1.$$

Notice that $\widetilde{SL(n,\mathbb{Z})}$ can be thought of as the set of paths in $SL(n,\mathbb{R})$ from the identity to any point of $SL(n,\mathbb{Z})$, homotopic paths being identified.

Let us now prove the following lemmas:

Lemma 1.0: *If $g \in \widetilde{SL(2,\mathbb{Z})}$ is a product of commutators and is taken to the identity by the projection to $SL(2,\mathbb{Z})$ then g is the identity.*

Lemma 1.1: *For $n > 2$, let $s \in \widetilde{SL(n,\mathbb{Z})}$ be the image of the non-zero element of $\mathbb{Z}/(2)$ in $\widetilde{SL(n,\mathbb{Z})}$; s is a commutator.*

Proof of Lemma 1.1:

Consider $i, j \in \widetilde{SL(n,\mathbb{Z})}$ given by the following paths from $[0,1]$ to $SL(n,\mathbb{R})$:

$$i(t) = \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) & 0 & \dots & 0 \\ \sin(\pi t) & \cos(\pi t) & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad j(t) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \cos(\pi t) & -\sin(\pi t) & \dots & 0 \\ 0 & \sin(\pi t) & \cos(\pi t) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

It is a simple matter to check that $[i,j]=s$. ∎

Proof of Lemma 1.0:

It is well known that $SL(2,\mathbb{Z})=\mathbb{Z}/(4)*_{\mathbb{Z}/(2)}\mathbb{Z}/(6)$, where the right hand side denotes an amalgamated product. More precisely, the generators of $\mathbb{Z}/(4)$, $\mathbb{Z}/(6)$ and $\mathbb{Z}/(2)$ are, respectively,

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\quad\text{and}\quad \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

(it suffices to check that these matrices satisfy no extra relations: in order to see this, consider the action of $PSL(2,\mathbb{Z})$ over the upper half plane by Möbius functions; a fundamental domain is $-1/2\le\Re(z)\le 1/2$, $|z|\ge 1$). We have therefore the following presentations of $SL(2,\mathbb{Z})$ and $\widetilde{SL(2,\mathbb{Z})}$ in terms of generators and relations:

$$SL(2,\mathbb{Z})=\langle a,b|a^2=b^3,a^6=e\rangle,\quad \widetilde{SL(2,\mathbb{Z})}=\langle a,b|a^2=b^3\rangle.$$

It is now clear that the abelianizations of $SL(2,\mathbb{Z})$ and $\widetilde{SL(2,\mathbb{Z})}$ are $\mathbb{Z}/(12)$ and $\mathbb{Z}$, respectively, and that when the natural projection from $\widetilde{SL(2,\mathbb{Z})}$ to $SL(2,\mathbb{Z})$ is carried to the abelianizations we have the obvious projection of $\mathbb{Z}$ onto $\mathbb{Z}/(12)$. We thus see that the image of $\mathbb{Z}$ in the exact sequence above is taken injectively to the multiples of 12 in $\mathbb{Z}$, the abelianization of $\widetilde{SL(2,\mathbb{Z})}$. The statement of Lemma 1.0 follows directly from the injectivity of this map. ∎

Let us close this section by drawing a comparison between the constructions we saw in the last paragraph and the analogous constructions for $SL(n,\mathbb{Z})$, $n>2$. The abelianization of $SL(n,\mathbb{Z})$ is $\mathbb{Z}/(2)$: this follows from the fact that $SL(n,\mathbb{Z})$ is generated by matrices equal to the identity except for one entry equal to 1 or -1 outside the main diagonal. It is clear that all these generators are conjugate to one another and to their own inverses (but this is not at all true for $n=2$). For similar reasons, the abelianization of $\widetilde{SL(n,\mathbb{Z})}$ is also equal to $\mathbb{Z}/(2)$ and the natural projection from $\widetilde{SL(n,\mathbb{Z})}$ to $SL(n,\mathbb{Z})$ is taken to the identity map in the abelianization. The image of $\mathbb{Z}/(2)$ in the exact sequence above is therefore mapped to the identity in the abelianization of $\widetilde{SL(n,\mathbb{Z})}$. This shows that $s\in\widetilde{SL(n,\mathbb{Z})}$ as defined in the statement of Lemma 1.1 is a product of commutators, which, of course, only confirms what we already saw directly.

2. Affine structures

We begin by proving the result we most directly need:

Lemma 2.0: *Let X be a fibre bundle over M with coherently oriented fibres $\mathbb{T}^2$. Then it is possible to give smoothly varying affine stuctures to the fibres of X.*

Proof:

Give X a riemannian metric and consider the metric restricted to fibres. By ignoring scalar factors this gives us a conformal, or, given the orientation, complex structure on each fibre. By the uniformization theorem ([A]), each fibre is the quotient of the complex plane $\mathbb{C}$ by a lattice, thus providing us with the desired affine structure. ∎

This proof does not generalize to any other dimension. The next lemma shows that, as mentioned in the introduction, this kind of question is directly related to the homotopy type of the group of diffeomorphisms of $\mathbb{T}^n$.

Let G_d^n be the group of orientation preserving diffeomorphisms of $\mathbb{T}^n$ and let $G_a^n \subset G_d^n$ be the group of orientation preserving affine diffeomorphisms. The inclusion of G_a in G_d induces homomorphisms from $\pi_k(G_a)$ to $\pi_k(G_d)$, $k \geq 0$. Given a base point x_0 for $\mathbb{T}^n$, we can construct a continuous map from G_d^n to G_a^n fixing all points of G_a^n: just take an arbitrary diffeomorphism f to the unique affine diffeomorphism $\tilde{f}$ which is homotopic to f and takes x_0 to $f(x_0)$. Thus, the homomorphism induced by the inclusion is injective and we just consider $\pi_k(G_a)$ to be a subgroup of $\pi_k(G_d)$.

Lemma 2.1: *If for all $k \geq 0$ we have $\pi_k(G_d^n) = \pi_k(G_a^n)$ then any fibre bundle X over any CW-complex M with oriented fibres $\mathbb{T}^n$ admits a smoothly varying affine stucture on fibres. On the other hand, if $\pi_{k_0}(G_d) \neq \pi_{k_0}(G_a)$ there exists a smooth fibre bundle X over the sphere S^{k_0+1} with coherently oriented fibre $\mathbb{T}^n$ which admits no affine structure.*

Given the last lemma, Lemma 2.0 follows from the fact that G_a is a deformation retract of G_d for $n = 2$. This last fact is well known (see [EE]); it can be proven by the same method of going from a metric to an affine structure used in the proof of Lemma 2.0. Indeed, choose a base point on the torus and, given a diffeomorphism, consider the usual metric in the domain and the induced metric in the image. Take the obvious path deforming this metric into the usual one. Now use each metric to define a complex structure, an affine structure and finally the unique diffeomorphism affine with respect to the said structure taking the base point to its image by the original diffeomorphism.

It is also known that not all diffeomorphisms of $\mathbb{T}^n$ are isotopic to affine diffeomorphisms (i. e., $\pi_0(G_d^n) \neq \pi_0(G_a^n)$) for $n \geq 6$ ([??]) and that, for $n \geq 7$, the connected components of the identity in G_a and G_d are *not* homotopy equivalent (see [ABK], Theorem C). The statement corresponding to Lemma 2.0 for $\mathbb{T}^n$ is therefore *false* for $n \geq 6$.

Proof of Lemma 2.1:

Let A^n be the space of all affine structures of $\mathbb{T}^n$. The fibre bundle X yields another fibre bundle Y over M, with fibre A^n: the set of affine structures over fibres of X. Clearly, X admits a smoothly varying affine stucture on fibres iff Y admits a smooth section.

We claim that G_d^n is homeomorphic to $G_a^n \times A^n$. Indeed, fix a base point x_0 and a canonical affine structure $\mathbf{a}$ on $\mathbb{T}^n$. We associate to a given diffeomorphism f the affine diffeomorphism $\tilde{f}$ as above and $f(\mathbf{a})$, the affine structure induced on the image by $\mathbf{a}$ (in the domain) and f. This map is a bijection since we can recover f from $\tilde{f}$ and $f(\mathbf{a})$: f is the only diffeomorphism which is homotopic to $\tilde{f}$, satisfies $f(x_0) = \tilde{f}(x_0)$ and is *affine* from $\mathbf{a}$ to $f(\mathbf{a})$. Thus, $\pi_k(G_d^n) = \pi_k(G_a^n) \times \pi_k(A^n)$.

If $\pi_k(G_d^n) = \pi_k(G_a^n)$ for all k we have $\pi_k(A^n) = 1$ for all k. We use the cell structure of M in order to construct a section of Y. First choose arbitrary points (affine structures) above the 0-cells. Since A^n is path connected ($\pi_0(A^n) = 1$) we extend the section to the 1-cells. More generally, since $\pi_k(A^n) = 1$, we may extend the section to the $k+1$-cells and we are done.

Conversely, if $\pi_{k_0}(G_d) \neq \pi_{k_0}(G_a)$, consider a map from $\mathbb{S}^{k_0}$ to G_d which can not be deformed to a map from $\mathbb{S}^{k_0}$ to G_a. Use this map to glue along the boundaries two trivial $\mathbb{T}^n$ bundles over the closed unit balls of dimension $k_0 + 1$. The bundle so constructed admits no affine structure since such a structure would give us a homotopy of our map to a map with image contained in G_a, contradicting the hypothesis. ∎

3. Proof of Proposition A

An action as described in the Proposition is clearly sufficient to give an affine structure to the bundle: actions of elements of $\mathbb{R}^n$ are translations. The existence of an affine structure on the torus bundle is therefore a necessary condition for the existence of a smooth locally free action of $\mathbb{R}^n$ with orbits equal to the fibres.

Consider a fibre bundle X, with fibre $\mathbb{T}^n$ and base M. From such a fibre bundle we can construct in a natural way a vector bundle Y with fibre $\mathbb{R}^n$ and base M as follows: the fibre of Y over a point x of M is $H_1(T_x, \mathbb{R})$, where T_x is the fibre of X over x. If the fibres of X have an affine structure, an equivalent construction can be given: consider the tangent bundle to X, consider then the subbundle of vectors tangent to fibres and finally identify two vectors over different points in the same fibre (in X) if one is obtained from the other by translation: this gives us again the fibre bundle Y. Equivalently, sections of Y correspond to vector fields over X tangent to fibres and invariant by translation.

The affine structure induced by an action is such that the action (or the vector fields which describe it) is constant on fibres. On the other hand, given any action over an affine bundle, it is easy to construct another which is constant on fibres: just linearize it on each fibre. More precisely, the linearized action takes the isotropy group of a point on a fibre F to $H_1(F, \mathbb{Z})$ in the obvious way. There is, therefore, no loss of generality in considering only actions which are constant on fibres.

If X is affine, finding an action over X constant on fibres is equivalent to finding a base for each fibre of Y in a smooth way. In other words, the desired action exists if and only if Y is trivial as a vector bundle. This finishes the proof of Proposition A. ∎

4. Proof of Theorem B

We are now ready to prove Theorem B. From previous sections, we are concerned with the triviality of Y as a vector bundle. Notice that inside each fibre of Y lies the lattice $H_1(T_x, \mathbb{Z})$: Y can therefore be given the structure group $SL(n, \mathbb{Z})$. The question we have to answer is therefore: when is a vector bundle with structure group $SL(n, \mathbb{Z})$ trivial as a vector bundle with structure group $SL(n, \mathbb{R})$?

For purely expository reasons, we break up the theorem into three lemmas.

Lemma 4.0: *Let X be an oriented fibre bundle with oriented fibre $\mathbb{T}^2$ and base space M^2, an oriented compact surface. Then, there exists a smooth locally free action of $\mathbb{R}^2$ over X such that the orbits coincide with the fibres.*

Proof:

Since M is a compact orientable surface, we can think of M^2 as a CW-complex with one 0-cell, a few 1-cells and one 2-cell in the usual way. Let us consider the vector bundle Y as above: is it trivial? Well, we can certanly define a basis for the fibres on the 0-cell; we can even assume that we took a basis of the canonical copy of $\mathbb{Z}^2$ in it. The connectedness of $SL(2, \mathbb{R})$ shows that it is just as easy to extend this definition to the 1-skeleton. Is it possible to extend this definition to the 2-cell? As we shall see, in our case the answer is 'yes'. We can use the canonical $\mathbb{Z}^2$ to define a temporary basis on the 2-cell, and think that we must apply some element of $SL(2, \mathbb{R})$ to each fibre so that it will match what was already defined on the 1-skeleton. This element is predetermined on the boundary, so the question is whether the loop defined by the boundary in $SL(2, \mathbb{R})$ is nullhomotopic, that is, whether it corresponds to the identity in $\widetilde{SL(2, \mathbb{R})}$. But each 1-cell gives us an element of $\widetilde{SL(2, \mathbb{Z})}$, and the 1-cells are disposed in such a way that the loop will be a product of commutators of elements of $\widetilde{SL(2, \mathbb{Z})}$. On the other hand the whole loop obviously maps to the identity in $SL(2, \mathbb{Z})$. By Lemma 1.0, Lemma 4.0 follows. ∎

The reader should notice that the above proof *does* work for $\mathbb{S}^2$, albeit in a somewhat trivial and degenerate way.

Lemma 4.1: *Let $n > 2$. There exists an oriented fibre bundle X with oriented fibre $\mathbb{T}^n$ and base space $\mathbb{T}^2$ such that there exists no smooth locally free action of $\mathbb{R}^n$ over X such that the orbits coincide with the fibres.*

Proof:

The reasoning above (which is, of course, no longer valid) can be used as a guideline to build a counter-example. First consider $[0, 1] \times [0, 1] \times \mathbb{T}^n$; let us now glue the sides in the following way: identify $(1, y, z_1, z_2, z_3, \dots, z_n)$ with $(0, y, -z_1, -z_2, z_3, \dots, z_n)$ and $(x, 1, z_1, z_2, z_3, \dots, z_n)$ with $(x, 0, z_1, -z_2, -z_3, \dots, z_n)$. The construction in the previous lemma would fail for this bundle exactly because following the boundary we see $[i, j] = s \neq e$.

We can certify ourselves of the fact that there is indeed no action by observing that $w_2(Y) \neq 0$, where $w_2(Y) \in H^2(\mathbb{T}^2, \mathbb{Z}/(2)) = \mathbb{Z}/(2)$ is the second Stiefel-Whitney class (see

[MS]). The computation of w_2 is easy: write $Y = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_\star$ where Y_1 corresponds to the direction z_1, Y_2 to z_2, Y_3 to z_3 and $Y_\star$ to the other directions. We have $w(Y_1) = 1 + e_1$, $w(Y_2) = 1 + e_2$, $w(Y_3) = 1 + e_1 + e_2$ and $w(Y_\star) = 1$, where e_1 and e_2 are the canonical generators of $H^1(\mathbb{T}^2, \mathbb{Z}/(2))$; by the Whitney product theorem, $w(Y) = 1 + e_1 \cup e_2$, where $e_1 \cup e_2$ is the non-zero element of $H^2(\mathbb{T}^2, \mathbb{Z}/(2))$. ∎

Lemma 4.2: *Let $m \geq 2$. There exists a fibre bundle X with oriented fibre $\mathbb{T}^2$ and base space $\mathbb{P}^m$ such that there exists no smooth locally free action of $\mathbb{R}^2$ over X such that the orbits coincide with the fibres.*

Remember that the real projective space $\mathbb{P}^m$ is orientable for m odd; the cases $m = 2$ and $m = 3$ therefore provide us with the two counter-examples necessary for the completion of the proof of Theorem B.

Proof:

Let Y_1 be the canonical line bundle over $\mathbb{P}^m$ and let $Y = Y_1 \oplus Y_1$ be the Whitney sum of two identical bundles (notice that fibres of Y are orientable). Assume that we have the usual Riemannian metric (smoothly) defined on the fibres of Y_1 and call the two unit vectors on each fibre $\pm v$. Let u_1 and u_2 be vectors in Y with the same basepoint: we say $u_1 \equiv u_2$ iff $u_1 - u_2 \in \mathbb{Z}v \oplus \mathbb{Z}v$; this definition clearly does not depend on the choice of v. Let X be the quotient of Y by this equivalence relation: X is the desired fibre bundle. The fact that no action exists follows from $w_2(Y) \neq 0$, where $w_2(Y) \in H^2(\mathbb{P}^n, \mathbb{Z}/(2)) = \mathbb{Z}/(2)$. Indeed, $w(Y_1) = 1 + e$ where e is the non-zero element of $H^1(\mathbb{P}^n, \mathbb{Z}/(2))$ and therefore $w(Y) = 1 + e^2$. ∎

References:

[A] L. V. Ahlfors: *Conformal Invariants, Topics in Geometric Function Theory*, McGraw-Hill Book Company (1973).

[ABK] P. L. Antonelli, D. Burghelea and P. J. Kahn: *The non-finite homotopy type of some diffeomorphism groups*, Topology 11 (1970), 1-49.

[EE] J. Eells and C. J. Earle: *A fibre bundle description of Teichmüller theory*, J. Diff. Geom. 3 (1969), 19-43.

[MS] J. W. Milnor and J. D. Stasheff: *Characteristic Classes*, Annals of Mathematical Studies, N. 76, Princeton University Press.

Nicolau C. Saldanha
Instituto de Matematica Pura e Aplicada
Estrada Dona Castorina, 110
Jardim Botânico, Rio de Janeiro, RJ 22460-320, BRAZIL
nicolau@impa.br; http://www.impa.br/~nicolau/