Nilpotent pseudogroups of functions on an interval

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Abstract

A *near-identity* nilpotent pseudogroup of order $m \geq 1$ is a family $f_1, \ldots, f_n : (-1, 1) \to \mathbb{R}$ of C^2 functions for which: $|f_i - \mathrm{id}|_{C^1} < \epsilon$ for some small positive real number $\epsilon < 1/10^{m+1}$ and commutators of the functions f_i of order at least m equal the identity. We present a classification of near-identity nilpotent pseudogroups: our results are similar to those of Plante, Thurston, Farb and Franks. As an application, we classify certain foliations of nilpotent manifolds.

Keywords: Nilpotent, pseudogroup, foliations, functions on an interval.

MSC-class: 37E05; 37E45; 57S25; 57R30

1 Introduction

A *near-identity* nilpotent pseudogroup of order $m \geq 1$ is a family $f_1, \ldots, f_n : (-1, 1) \to \mathbb{R}$ of C^2 functions for which:

- $|f_i - \mathrm{id}|_{C^1} < \epsilon$ for some small positive real number $\epsilon < 1/10^{m+1}$;
- commutators of the functions f_i of order at least m equal the identity.

More precisely, the pseudogroup is generated by the family of functions but we shall often make the abuse of confusing these two objects. A commutator of order 1 is of the form $[f_i, f_j]$ and, for $m > 1$, a commutator of order m is a function of the form $[g_1, g_2] = g_1^{-1} \circ g_2^{-1} \circ g_1 \circ g_2$ where g_1 is one of the original functions or a commutator of order less than m and g_2 is a commutator of order $m - 1$. In particular, a family of functions which commute is a nilpotent pseudogroup of order 1: we call this an *abelian* pseudogroup. If all commutators commute with each other, then we call the pseudogroup *metabelian.* A common fixed point for the pseudogroup is a point x for which $f_i(x) = x$ for all i.

If the functions f_i are bijections from $(-1, 1)$ to itself, then such a nilpotent pseudogroup is just a group of functions. Plante and Thurston prove that nilpotent groups of diffeomorphisms on $[-1, 1)$ or $(-1, 1]$ are abelian ([8]). Farb and Franks consider groups of diffeomorphisms of $(-1, 1)$ and prove several results, among them that $\mathit{Diff}^\infty((-1, 1))$ contains nilpotent subgroups of arbitrary degree of nilpotency but that such subgroups are all metabelian ([5]). Our work differs from these in that we consider pseudogroups instead of groups: the main new difficulty is that long compositions of the functions f_i may not define any function since the domain may vanish. More precisely, we prove the following theorem.

Theorem 1 *Any near-identity nilpotent pseudogroup of functions $f_1, \ldots, f_n$ is metabelian. Furthermore, any near-identity pseudogroup fits into one of the three cases below.*

1. *There exists at least one common fixed point and the pseudogroup is abelian. Furthermore, for each maximal interval $I_1 \subset (-1, 1)$ containing no common fixed points, there exist real constants a_i and an increasing homeomorphism $\phi : J \to I_1$, $J \subseteq \mathbb{R}$ with $f_i(\phi(t)) = \phi(t + a_i)$ whenever $t, t + a_i \in J$. If $\inf I_1 > -1$ (resp. $\sup I_1 < 1$) then $\inf J = -\infty$ (resp. $\sup J = +\infty$).*

2. *There exists no common fixed point, the pseudogroup is abelian, there exist real constants a_i and an increasing homeomorphism $\phi : J \to (-1, 1)$, $J \subseteq \mathbb{R}$ with $|J| > |a_i|$ and $f_i(\phi(t)) = \phi(t + a_i)$ whenever $t, t + a_i \in J$.*

3. *There exists no common fixed point, there exist integer constants a_i and a finite set $\{y_{-N}, y_{-N+1}, \ldots, y_N\} \subset (-1, 1)$ with $y_i < y_{i+1}$, $y_{-N} < -1 + \epsilon$, $y_N > 1 - \epsilon$, $N > |a_i|$, such that $f_i(y_k) = y_{k+a_i}$.*

This subject was motivated by the study of actions of nilpotent Lie groups on manifolds. In particular, we study actions of the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$$

on manifolds of dimension 4 ([1], [2]). We do not discuss actions here but we present an application of theorem 1 to foliations.

We consider compact orientable manifolds of the form $G/H = \{gH, g \in G\}$, where G is a nilpotent Lie group and $H = \pi_1(G/H) \subset G$ is a discrete cocompact subgroup. We may assume that G/H has a smooth Riemann metric and therefore

$G/H \times (-1,1)$ also has a metric. The foliation $\mathcal{F}_0$ of $G/H \times (-1,1)$ with leaves of the form $G/H \times \{x\}$ is called *horizontal*; this foliation can be defined as being perpendicular to the vertical vector field Z_0 at any point of $G/H \times (-1,1)$. An arbitrary transversally orientable foliation $\mathcal{F}_1$ of $G/H \times (-1,1)$ of codimension 1 can be similarly described as being perpendicular to some unit vector field Z_1 at any point; we define $d_{C^1}(\mathcal{F}_0, \mathcal{F}_1) = d_{C^1}(Z_0, Z_1)$. Let $\gamma : [0,1] \to G/H$ be a smooth path; if $d_{C^1}(\mathcal{F}_0, \mathcal{F}_1)$ is sufficiently small then, for $x \in (-1+\epsilon, 1-\epsilon)$, the path γ can be *lifted* to a unique smooth path $\gamma_x : [0,1] \to G/H \times (-1,1)$ with $\gamma_x(0) = (\gamma(0), x)$, $\gamma_x(t) = (\gamma(t), *)$ and $\gamma'_x(t)$ tangent to $\mathcal{F}_1$ for all t ([3]). This defines a function $f_\gamma : (-1+\epsilon, 1-\epsilon) \to (-1,1)$ taking x to the second coordinate of $\gamma_x(1)$: if γ is a generator of $\pi_1(G/H)$, f_γ is called the *holonomy* of $\mathcal{F}_1$. If $\gamma_1, \gamma_2, \ldots, \gamma_n$ are generators of $\pi_1(G/H)$ and $d_{C^1}(\mathcal{F}_0, \mathcal{F}_1)$ is sufficiently small then $f_{\gamma_1}, f_{\gamma_2}, \ldots, f_{\gamma_n}$ form a near-identity nilpotent pseudogroup. We call a foliation of $G/H \times (-1,1)$ *abelian* (resp. *metabelian*) if its pseudogroup is abelian (resp. metabelian). Similarly, we call a leaf of a foliation *abelian* if $\gamma_x(1) = \gamma_x(0)$ whenever $\gamma \in \pi_1(G/H)$ is a commutator and $\gamma_x(0)$ belongs to the leaf; thus, a foliation is abelian if and only if all its leaves are abelian.

Let $\alpha : G \to \mathbb{R}$ be a group homomorphism: the 1-form $d\alpha$ can be lifted to G/H and therefore to $G/H \times \mathbb{R}$, where it is closed but probably not exact. The 1-form dx, where x is the second coordinate, is exact in $G/H \times \mathbb{R}$ and therefore $d\alpha + dx$ is a closed 1-form in $G/H \times \mathbb{R}$, defining a foliation $\mathcal{F}_\alpha$. Notice that $\mathcal{F}_\alpha$ is abelian.

Theorem 2 *Given a manifold of the form G/H there exists $\epsilon_{G/H}$ such that, if $\mathcal{F}_1$ is a smooth foliation of $G/H \times (-1,1)$ with $d_{C^1}(\mathcal{F}_0, \mathcal{F}_1) < \epsilon_{G/H}$, then $\mathcal{F}_1$ is metabelian and one of the three conditions hold:*

1. *$\mathcal{F}_1$ has at least one compact leaf, is abelian and, for any maximal connected open set M of $G/H \times (-1,1)$ containing no compact leaf, there exists a homomorphism $\alpha : G \to \mathbb{R}$, an open set $\mathcal{J} \subseteq G/H \times \mathbb{R}$ and a homeomorphism $\Phi : \mathcal{J} \to M$ taking $\mathcal{F}_\alpha$ to $\mathcal{F}_1$.*

2. *$\mathcal{F}_1$ has no compact leaf, is abelian and there exists a homomorphism $\alpha : G \to \mathbb{R}$, an open set $\mathcal{J} \subseteq G/H \times \mathbb{R}$ and a homeomorphism $\Phi : \mathcal{J} \to G/H \times (-1,1)$ taking $\mathcal{F}_\alpha$ to $\mathcal{F}_1$.*

3. *$\mathcal{F}_1$ has no compact leaf and has an abelian leaf closed in $G/H \times (-1,1)$, arriving or accumulating both at $G/H \times \{1\}$ and $G/H \times \{-1\}$.*

Some key results, presented in section 2, are that there exist nonempty closed sets X invariant under the pseudogroup of functions and such that the restrictions to X of all commutators equal the identity; in particular, $[f_i, f_j]$ has many fixed points. In section 3 we define a concept of *translation number* $\tau(f_i, f_j)$ for functions f_i, f_j in a near-identity nilpotent pseudogroup; this concept is also present in [5] and reduces to the usual definition of rotation number if $[f_i, f_j] = \mathrm{id}$. As we shall see in section 4, Denjoy's theorem implies that if some translation number is irrational then the invariant set X constructed in section 2 is an interval. In section 5 we use Koppel's lemma to prove that near-identity nilpotent pseudogroups are metabelian and finally, in section 6, we bring together the results in order to prove theorems 1 and 2.

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2 Fixed points

We recall the usual definition of translation number for continuous increasing functions $u : \mathbb{R} \to \mathbb{R}$ of degree one, i.e., with $u(x+1) = u(x) + 1$, or, more generally, for a continuous increasing function $u : [0,1] \to \mathbb{R}$ with $u(1) = 1 + u(0)$. Assume $0 < u(0) < 1$. Define the sequence $a_0 = 0$,

$$a_{n+1} = \begin{cases} u(a_n), & \text{if } u(a_n) < 1, \\ u(a_n) - 1, & \text{otherwise.} \end{cases}$$

The translation number $\tau(u)$ of u is the proportion of points of this sequence in the interval $[0, u(0))$. More precisely, define $p(0) = 0$, $p(n+1) = p(n) + 1$ if $0 \le a_n < u(0)$ and $p(n+1) = p(n)$ otherwise: then the limit

$$\lim_{n \to \infty} \frac{p(n)}{n}$$

exists and is called $\tau(u)$. Denjoy's theorem [4] states that if u is a degree one function of class C^2 with irrational translation number $\alpha = \tau(u)$ then there exists a homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, also of degree one, such that $u(\phi(t)) = \phi(t + \alpha)$ for all t.

The condition $|f_i - \mathrm{id}|_{C^1} < \epsilon < 1/10^{m+1}$ guarantees that f_i is a diffeomorphism from $(-1, 1)$ to some open interval I,

$$(-0.99, 0.99) \subset (-1+\epsilon, 1-\epsilon) \subset I \subset (-1-\epsilon, 1+\epsilon) \subset (-1.01, 1.01)$$

and we take $f_i^{-1} : I \cap (-1, 1) \to \mathbb{R}$. A composition such as the commutator

$$[f_i, f_j] = f_i^{-1} \circ f_j^{-1} \circ f_i \circ f_j$$

is defined in the largest possible domain such that intermediate expressions are in $(-1,1)$: this is clearly some interval I, $(-0.96,0.96) \subset I \subseteq (-1,1)$. An equality such as $[f_i,[f_j,f_k]] = \mathrm{id}$ means that both functions coincide in the intersection of their domains, in this case at least $(-1+10\epsilon, 1-10\epsilon)$. A set $X \subseteq (-1,1)$ is said to be invariant under the functions f_i if $x \in X$ implies that $f_i(x)$ and $f_i^{-1}(x)$ belong to $X \cup (-\infty,-1] \cup [1,\infty)$ for all i. Recall that $X \subseteq (-1,1)$ is closed in $(-1,1)$ if $\overline{X} \cap (-1,1) = X$.

Proposition 2.1 *For any near-identity nilpotent pseudogroup of functions there exists a nonempty set $X \subseteq (-1,1)$ which is closed in $(-1,1)$, invariant under the functions f_i and such that $[f_i,f_j]|_X = id$ for all i,j.*

If x is a common fixed point, $X = \{x\}$ is a trivial example of a closed invariant set as described in the proposition. We shall be more interested in nontrivial invariant sets X. Notice that if x_0 and x_1 are common fixed points then the proposition may be applied to the restriction of the functions to the interval (x_0,x_1): in other words, inside each maximal interval I in the complement of the set of common fixed points, there exists a nonempty invariant set $X_I \subseteq I$ closed in I and such that $[f_i,f_j]|_{X_I} = \mathrm{id}$ for all i,j.

This proposition will be proved by induction, the key step being the following lemma.

Lemma 2.2 *Let $f_1,\ldots,f_n$ be functions of class C^2, $|f_i - id|_{C^1} < \epsilon$ (where $\epsilon > 0$, $\epsilon < 1/100$). Assume there exists a set $X \neq \emptyset$ closed in $(-1,1)$ and invariant under the functions f_i such that $[f_i,[f_1,f_2]]|_X = id$ for all i. Then there exists $Y \subseteq X$, $Y \neq \emptyset$, Y also closed in $(-1,1)$ and invariant under the functions f_i such that $[f_1,f_2]|_Y = id$.*

Proof of Lemma 2.2: Let Y be the set of fixed points of $[f_1,f_2]$: the set Y is clearly closed and invariant under the functions f_i; it suffices to prove that $Y \neq \emptyset$.

If $X \cap (-1/2,1/2) = \emptyset$, let $x_0 \in X$ be the element of least absolute value; assume without loss of generality that $x_0 > 0$. We claim that $f_i(x_0) = x_0$ for all i: indeed, $f_i(x_0) < x_0$ would be another element of X of smaller absolute value whence $f_i(x_0) \geq x_0$. Similarly, $f_i^{-1}(x_0) \geq x_0$ and since $f_i' \geq 0$ the claim follows. We can now take $Y = \{x_0\}$ and this proves the lemma in this case; we assume from now on $X \cap (-1/2,1/2) \neq \emptyset$.

Let $x_0 \in X \cap (-1/2,1/2)$ be an arbitrary point. If $f_1(x_0) = f_2(x_0) = x_0$ we are done. Otherwise we may assume without loss of generality that $f_1^{-1}(x_0) < x_0 \leq f_2(x_0) \leq f_1(x_0)$, $x_0 < f_1(x_0)$. We prove that there exists a fixed point of the commutator $[f_1,f_2]$ in the interval $[f_1^{-1}(x_0), f_1(x_0)]$: this will prove the lemma.

Set $\delta = f_1(x_0) - x_0 > 0$, $I_k = (f_1^{-k}(x_0), f_1^k(x_0))$. Clearly $I_1 \subset I_2 \subset \cdots \subset I_{10} \subset (-3/4,3/4)$ and both $f_1(I_k)$ and $f_1^{-1}(I_k)$ are contained in I_{k+1} for $k > 0$, $k < 10$.

Notice that $\delta(1-\epsilon)^{|k|+1} < f_1^{k+1}(x_0) - f_1^k(x_0) < \delta(1+\epsilon)^{|k|+1}$ for $-10 \le k < 10$; in particular, $f_1^k(x_0) - x_0 < k(1+\epsilon)^k\delta$ for $k > 0$, $k < 10$.

We claim that $f_2(I_k)$ and $f_2^{-1}(I_k)$ are both contained in I_{k+2} for all $k > 0$, $k < 9$. From the mean value theorem, $f_2 f_1^k(x_0) - f_2(x_0) < (1+\epsilon)(f_1^k(x_0) - x_0)$ whence, adding $f_2(x_0) \le f_1(x_0)$, $f_2 f_1^k(x_0) < f_1^k(x_0) + (1 + \epsilon k(1+\epsilon)^k)\delta$. On the other hand, $f_1^{k+2}(x_0) > f_1^k(x_0) + 2(1-\epsilon)^{k+1}\delta$ and it follows that $f_2 f_1^k(x_0) < f_1^{k+2}(x_0)$ provided $2(1-\epsilon)^{k+1} > 1 + \epsilon k(1+\epsilon)^k$, which indeed happens for $\epsilon < 1/100$ and $k \le 8$. Similarly, $f_2^{-1} f_1^k(x_0) < f_1^{k+2}(x_0)$, $f_2 f_1^{-k}(x_0) > f_1^{-k-2}(x_0)$ and $f_2^{-1} f_1^{-k}(x_0) > f_1^{-k-2}(x_0)$, proving our claim.

Let g be the commutator $[f_1, f_2]$. From the claim above it follows easily that $g(I_k)$ and $g^{-1}(I_k)$ are both contained in I_{k+4} for $k > 0$, $k < 6$. For example, $f_2 f_1^k(x_0) < f_1^{k+2}(x_0)$ implies $f_1 f_2 f_1^k(x_0) < f_1^{k+3}(x_0)$ and $f_2^{-1} f_1 f_2 f_1^k(x_0) < f_1^{k+5}(x_0)$ whence, finally, $[f_1, f_2] f_1^k(x_0) = f_1^{-1} f_2^{-1} f_1 f_2 f_1^k(x_0) < f_1^{k+4}(x_0)$.

Let $\psi : I_{10} \to \mathbb{R}$ be a function with positive derivative of class C^2 satisfying $\psi(x_0) = 0$, $\psi(f_1(x)) = 1 + \psi(x)$: such a function can be contructed by taking a diffeomorphism from $[x_0, f_1(x_0)]$ to $[0,1]$ with compatible behavior at boundary points. Define $\tilde{f}_i : (-8,8) \to \mathbb{R}$ and $\tilde{g} : (-6,6) \to \mathbb{R}$ by $\tilde{f}_i = \psi f_i \psi^{-1}$ and $\tilde{g} = \psi g \psi^{-1}$. Notice that $\tilde{f}_1(x) = x+1$. Also, set $\tilde{X} = \psi(X \cap I_{10})$: if $x \in \tilde{X}$, $|x| < 6$ then the points $\tilde{f}_i(x)$, $\tilde{f}_i^{-1}(x)$, $\tilde{g}(x)$ and $\tilde{g}^{-1}(x)$ are all in $\tilde{X}$. Let $\breve{g} : \mathbb{R} \to \mathbb{R}$ be a function of degree 1 (i.e., $\breve{g}(x+1) = \breve{g}(x) + 1$ for all x) with $\breve{g}|_{\tilde{X}} = \tilde{g}|_{\tilde{X}}$. In order to prove the existence of such a function $\breve{g}$, we consider two cases. If $[0,1] \subseteq \tilde{X}$ then $\tilde{g}(x+1) = \tilde{g}(x) + 1$ for all $x \in (-5,5)$ and $\breve{g}$ is obtained by extending $\tilde{g}$. Otherwise, let $[x_1, x_2] \subset (0,1)$, $[x_1, x_2] \cap \tilde{X} = \{x_1, x_2\}$ and define $\breve{g}$ to be an arbitrary function of degree 1 coinciding with $\tilde{g}$ in the interval $[x_2 - 1, x_1]$. Let c be the translation number of $\breve{g}$: we already proved that $|c| < 4$; we claim that $c = 0$. The claim implies that the sequence $0, \breve{g}(0), \ldots, \breve{g}^k(0), \ldots$ converges to a fixed point $\psi(x_1)$ of $\breve{g}$ in $\tilde{X} \cap [-1,1]$ and x_1 is the desired fixed point of g, proving the lemma. In order to prove the claim, it is convenient to consider, by contradiction, the cases c irrational and c rational, $c \neq 0$.

Case 1: c irrational.

By Denjoy theorem there exists a homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(n) = n$ for $n \in \mathbb{Z}$ and

$$\hat{f}_1(x) = \phi \tilde{f}_1 \phi^{-1}(x) = x + 1, \quad \hat{g}(x) = \phi \breve{g} \phi^{-1}(x) = x + c.$$

Define $\hat{f}_2 : (-7,7) \to \mathbb{R}$ by

$$\hat{f}_2(x) = \phi \tilde{f}_2 \phi^{-1}(x).$$

We have $\hat{f}_2 \hat{f}_1 \hat{g} = \hat{f}_1 \hat{f}_2$ and $\hat{f}_2 \hat{g} = \hat{g} \hat{f}_2$ on $\hat{X} = \phi(\tilde{X})$ which become

$$\hat{f}_2(x+1+c) = \hat{f}_2(x) + 1, \qquad \hat{f}_2(x+c) = \hat{f}_2(x) + c, \qquad x \in \hat{X}.$$

Also, since $f_2(I_k) \subset I_{k+2}$ we have $\hat{f}_2((-k,k)) \subset (-k-2, k+2)$ for $k \leq 7$.

Consider the set A of points $(x,y) \in \mathbb{Z}^2$ with $|x(1+c)+yc| < 5$: this set is connected in the sense that points of A can be joined by a path with vertices in A and edges of size 1. Let (x_k, y_k), $k = 0, \ldots, N$ be such a path of points of A with $(x_0, y_0) = (0,0)$ and $|x_N + y_N c| > 9$. From $|x_k(1+c) + y_k c| < 5$ we have $|\hat{f}_2(x_k(1+c)+y_k c)| < 7$ for all $k \leq N$. If $(x_{k+1}, y_{k+1}) = (\pm 1, 0) + (x_k, y_k)$ then $\hat{f}_2(x_{k+1}(1+c) + y_{k+1}c) = \hat{f}_2(x_k(1+c)+y_k c) \pm 1$. Also, if $(x_{k+1}, y_{k+1}) = (0, \pm 1) + (x_k, y_k)$ then $\hat{f}_2(x_{k+1}(1+c)+y_{k+1}c) = \hat{f}_2(x_k(1+c)+y_k c) \pm c$. Thus $\hat{f}_2(x_k(1+c)+y_k c) = \hat{f}_2(0) + x_k + y_k c$ for all k. In particular $|x_N + y_N c| = |\hat{f}_2(x_N(1+c)+y_N c) - \hat{f}_2(0)| < 9$, a contradiction.

Case 2: c rational, $c \neq 0$.

Let $c = p/q$, $p, q \in \mathbb{Z}$, $q > 0$. Assume without loss of generality that $p > 0$. Let $h = \tilde{f}_1^{-p} \breve{g}^q$: we know that h has a fixed point $x_1 \in [0,1]$. Take $x_2 \in \tilde{X} \cap [x_1, x_1+1]$: the sequence $x_2, h(x_2), h^2(x_2), \ldots$ is contained in the compact set $\tilde{X} \cap [x_1, x_1+1]$ and therefore any accumulation point x_3 of this sequence is a fixed point of h in $\tilde{X}$.

Define $z_0 = x_3$ and

$$z_{i+1} = \begin{cases} \tilde{f}_1^{-1}(z_i), & z_i \geq x_3 + 1, \\ \breve{g}(z_i), & \text{otherwise.} \end{cases}$$

We have $z_{p+q} = z_0$ and in this sequence we take p times the first case and q times the second. Set $w_i = \tilde{f}_2^{-1}(z_i)$: we have therefore

$$w_{i+1} = \begin{cases} \breve{g}^{-1}\tilde{f}_1^{-1}(w_i), & w_i \geq f_2^{-1}(x_3+1), \\ \breve{g}(w_i), & \text{otherwise.} \end{cases}$$

Thus $\tilde{f}_1^{-p}\breve{g}^{q-p}$ has fixed point w_0 and the translation number of $\breve{g}$ is $p/(q-p)$, a contradiction. ∎

Proof of Proposition 2.1: We proceed by induction on m. Assume our pseudogroup of functions to be nilpotent of order m. Apply lemma 2.2 to the family of commutators of order at most $m-1$, $X_0 = (-1,1)$, with the new f_1 being an arbitrary commutator of order at most $m-1$ and the new f_2 being a commutator or order $m-1$. We thus obtain a closed invariant subset X_1 of X_0 where a given commutator of order m equals the identity. Repeating this process we obtain a closed invariant subsets $X_2 \supseteq X_3 \supseteq \cdots \supseteq X_k$ such that all commutators of order m equal the identity in X_k. Now apply the induction hypothesis to obtain $X \subseteq X_k$. ∎

If in the constructions performed in the proof of the lemma we take x_0 to be a fixed point of g then $f_1 f_2(x_0) = f_2 f_1(x_0)$ and $x_0 \leq f_2(x_0) \leq f_1(x_0)$ implies $f_1(x_0) \leq f_2 f_1(x_0) = f_1 f_2(x_0) \leq f_1^2(x_0)$ and $f_2(I_k) \subseteq I_{k+1}$. Also, $g(I_k) = I_k$.

3 Translation number

We modify the definition of translation number to define the translation number of f_2 relative to f_1, where both functions belong to a near-identity nilpotent pseudogroup. Let x_0 be a point of X, a nontrivial invariant set as discussed in the previous section. Assume that $f_1^{-1}(x_0) < x_0 \leq f_2(x_0) \leq f_1(x_0)$. Define a sequence of points starting at $a_0 = x_0$ by $a_{n+1} = f_1^{-k(n)} f_2(a_n)$ where $k(n)$ is the only integer for which $x_0 \leq f_1^{-k(n)} f_2(a_n) < f_1(x_0)$. Since $f_1(x_0) \leq f_1 f_2(x_0) = f_2 f_1(x_0) \leq f_1^2(x_0)$ and, by contruction, $x_0 \leq a_n < f_1(x_0)$ we have $x_0 \leq f_2(x_0) \leq f_2(a_n) < f_2 f_1(x_0) \leq f_1^2(x_0)$ and therefore $k(n)$ is 0 or 1. Also, $k(n) = 1$ if and only if $x_0 \leq a_{n+1} < f_2(x_0)$. Let $F_0^{x_0} = \mathrm{id}$ and $F_{n+1}^{x_0} = f_1^{-k(n)} \circ f_2 \circ F_n^{x_0}$: as usual, we must show that this composition makes sense in a large domain. As in the proof of proposition 2.1, let $I_n = (f_1^{-n}(x_0), f_1^n(x_0))$; notice that I_n is well defined at least for $n \leq 10$ and that $f_2(I_n) \subseteq I_{n+1}$.

Lemma 3.1 *For any positive integer n, $F_n^{x_0}$ is well defined in I_8 and $F_n^{x_0}(I_8) \subseteq I_9$. Also,*

$$f_1^{-8}(x_0) \leq F_n^{x_0} f_1^{-8}(x_0) = f_1^{-8} F_n^{x_0}(x_0) < f_1^{-7}(x_0),$$

$$f_1^{8}(x_0) \leq F_n^{x_0} f_1^{8}(x_1) = f_1^{8} F_n^{x_0}(x_1) < f_1^{9}(x_0).$$

Moreover, X is invariant under $F_n^{x_0}$.

Proof: We prove the inequalities in the statement by induction on n, the case $n = 1$ being easy. By definition, $F_{n+1} f_1^{\pm 8}(x_0) = f_1^{-k(n)} f_2 F_n f_1^{\pm 8}(x_0)$: this already shows that these two expressions make sense since by induction hypothesis both $F_n f_1^{\pm 8}(x_0)$ make sense and are in I_9. By the induction hypothesis, $F_{n+1} f_1^{\pm 8}(x_0) = f_1^{-k(n)} f_2 f_1^{\pm 8} F_n(x_0)$. Since $F_n(x_0) \in X$, $F_{n+1} f_1^{\pm 8}(x_0) = f_1^{\pm 8} f_1^{-k(n)} f_2 F_n(x_0) = f_1^{\pm 8} F_{n+1}(x_0)$. The other claims are now easy. ■

Set $p(0) = 0$, $p(n+1) = p(n) + k(n)$: we define the translation number to be the limit

$$\tau(f_2, f_1, x_0) = \lim_{n \to \infty} \frac{p(n)}{n};$$

we still have to prove that this limit exists. If $f_2^{-1}(x_0) < x_0 \leq f_1(x_0) \leq f_2(x_0)$ we can make a similar construction reverting the roles of f_1 and f_2 and define

$$\tau(f_2, f_1, x_0) = 1/(\tau(f_1, f_2, x_0));$$

if $f_1^{-1}(x_0) \leq f_2(x_0) \leq x_0 < f_1(x_0)$ we define

$$\tau(f_2, f_1, x_0) = -\tau(f_2^{-1}, f_1, x_0);$$

the other cases are similar.

Let Z be the closed set of common fixed points of all functions f_i. We show that the translation number is well defined in each connected component of the complement of Z.

Proposition 3.2 *If $x_0, x_1 \in X$ are in the same connected component of the complement of Z then $\tau(f_2, f_1, x_0) = \tau(f_2, f_1, x_1)$.*

Proof: Let $g = [f_1, f_2]$. We first prove that the limit exists. Assume $f_1^{-1}(x_0) < x_0 \leq f_2(x_0) \leq f_1(x_0)$ and let ψ be a conjugation between f_1 and $x \mapsto x+1$ so that $\tilde{f}_1(x) = \psi f_1 \psi^{-1}(x) = x + 1$, $\tilde{f}_2 = \psi f_2 \psi^{-1}$ and $\tilde{g} = \psi g \psi^{-1}$; $\psi : I_{10} \to \mathbb{R}$ is a function with positive derivative of class C^2 and $\psi(x_0) = 0$ as constructed in the proof of Lemma 2.2. Notice that $\tilde{f}_2(x+1) = \tilde{f}_2(x) + 1$ if and only if $\tilde{g}(x) = x$; in particular $\tilde{f}_2(1) = \tilde{f}_2(0) + 1$. The definition of translation number applies to the restriction to $[0, 1]$ of $\tilde{f}_2$; equivalently, let $\check{f}_2$ be the only function of degree 1 coinciding with $\tilde{f}_2$ in the interval $[0, 1]$:

$$\check{f}_2(x) = \tilde{f}_2(x - \lfloor x \rfloor) + \lfloor x \rfloor.$$

The points a_n constructed as above from f_1 and f_2 are all in the interval $[x_0, f_1(x_0))$ and $\psi(a_n)$ is always in the interval $[0, 1]$. The construction of $k(n)$, F_n and $p(n)$ only considers values of f_2 in the interval $[x_0, f_1(x_0))$, or, equivalently, values of $\tilde{f}_2$ in the interval $[0, 1)$. It makes therefore no difference whether we take $\tilde{f}_2$ or $\check{f}_2$ and $\tau(f_2, f_1, x_0)$ is the usual translation number of $\check{f}_2$.

Let x_1 be another fixed point of g, $x_0 \leq x_1 \leq f_1(x_0)$. The translation number of $\check{f}_2$ is the same if computed in the interval $[0, 1]$ or in the interval $[\psi(x_1), \psi(x_1) + 1]$. Furthermore, the functions $\tilde{f}_2$ and $\check{f}_2$ coincide in the orbit of x_1 since these are all fixed points of $\tilde{g}$ and the construction of the translation number coincides for these two functions. Thus $\tau(f_2, f_1, x_0) = \tau(f_2, f_1, x_1)$.

Let $x_0 < x_*$ be two fixed points of g in the same connected component of the complement of Z. Let $\epsilon > 0$ be the infimum over the compact interval $[x_0, x_*]$ of the positive continuous function $\max\{|f_1(x) - x|, |f_2(x) - x|\}$ and take a sequence $y_0 = x_0, y_1, \ldots, y_N = x_*$ with $0 < y_{i+1} - y_i < \epsilon/4$. Let x_i be the fixed point of g which is closest to y_i so that $x_0 \leq x_1 \leq \cdots \leq x_N = x_*$ are fixed points of g. We claim that $x_{i+1} \leq \max\{f_1(x_i), f_2(x_i), f_1^{-1}(x_i), f_2^{-1}(x_i)\}$. Assume without loss of gererality that the largest among these four numbers is $f_1(x_i)$: the interval $((x_i + f_1(x_i))/2, f_1(x_i))$ has size at least $\epsilon/2$ and therefore there is some y_j in it. The point $f_1(x_i)$ is a fixed point of g thus the distance between x_j and y_j is no larger than that between $f_1(x_i)$ and y_j: it follows that $x_i \leq x_{i+1} \leq x_j \leq f_1(x_i)$, as claimed. The previous paragraph can now be used to show that $\tau(f_2, f_1, x_i) = \tau(f_2, f_1, x_{i+1})$ and we have $\tau(f_2, f_1, x_0) = \tau(f_2, f_1, x_N)$.

■

4 The irrational case

We already saw that commutators in a near-identity nilpotent pseudogroup of functions have many fixed points; we now show that in many cases all commutators equal the identity so that the original functions commute.

Proposition 4.1 *Let f_1, f_2 be functions in a near-identity nilpotent pseudogroup and let X be a closed invariant subset where all the f_i commute. Let $x_0 \in X$, x_0 not a fixed point of f_1. If $\tau(f_2, f_1, x_0)$ is irrational then x_0 is an interior point of X.*

Proof: Assume without loss of generality that $f_1^{-1}(x_0) < x_0 \leq f_2(x_0) \leq f_1(x_0)$. Construct $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{g}$ and $\tilde{X}$ as usual where $g = [f_1, f_2]$ so that $\tilde{f}_1(x) = x + 1$; notice that $\tilde{g}(0) = 0$. Assume by contradiction that $(x_1, x_2) \subseteq (0, 1)$ is an open interval such that $[x_1, x_2] \cap \tilde{X} = \{x_1, x_2\}$. We shall construct a counter-example to Denjoy's theorem, thus obtaining a contradiction.

Let $\check{f}_2 : \mathbb{R} \to \mathbb{R}$ be an increasing function of degree 1 and class C^2 with $\check{f}_2(x) = \tilde{f}_2(x)$ for $x_2 - 1 < x < x_1$. The function $\check{f}_2$ is defined arbitrarily in the interval (x_1, x_2) with the only restrictions that it must be of class C^2, increasing, satisfy $\check{f}_2(x) = \tilde{f}_2(x)$ for x near x_1 and $\check{f}_2(x) = 1 + \tilde{f}_2(x - 1)$ for x near x_2; this is clearly possible. Now $\check{f}_2$ is a function of degree 1 and class C^2 and irrational translation number and $\tilde{X} + \mathbb{Z}$ is a nontrivial invariant closed set, contradicting Denjoy's theorem. ■

We may bring together our conclusions as a proposition.

Proposition 4.2 *Let $f_1, \ldots, f_n$ be a near-identity nilpotent pseudogroup of functions and let $x_0 \in (-1/2, 1/2)$. Then one of the following three situations holds:*

1. *x_0 is a common fixed point of the functions f_i.*

2. *There exists real constants a_i, an open interval $I \subseteq (-1, 1)$ containing $x_0, f_i(x_0), f_i^{-1}(x_0)$, and a homemorphism $\phi : J \to I$, $J \subseteq \mathbb{R}$ with $f_i(\phi(t)) = \phi(t + a_i)$ whenever $t, t + a_i \in J$.*

3. *There exist integer constants a_i, a finite set $\{y_{-N}, y_{-N+1}, \ldots, y_N\}$ with $y_i < y_{i+1}$ and $y_0 \leq x_0 < y_1$, $N > |a_i|$ with $f_i(y_k) = y_{k+a_i}$.*

Proof: If x_0 is not a common fixed point of the functions f_i then we may apply the results of the previous sections. If at least one translation number is irrational then we apply proposition 4.1 and we are in the second case. Otherwise the functions from $\mathbb{S}^1$ to itself constructed above all have rational translation numbers and therefore admit periodic points and we are in the third case. ■

5 Near-identity nilpotent pseudogroups are metabelian

We first state Koppel's lemma ([7]), an important result also in the works of Plante, Thurston, Farb and Franks.

Lemma 5.1 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be increasing diffeomorphisms, f of order C^2 and g of order C^1 with $[f, g] = id$. Assume that there exists a nondegenerate bounded open interval $I = (a, b)$ for which $f(b) = g(b) = b$, $f(a) > g(a) = a$ and $x \in (a, b)$ implies $f(x) > x$. Then $g(x) = x$ for all $x \in I$.*

Let $f_1, f_2, \ldots, f_n$ be a near-identity nilpotent pseudogroup with rational translation numbers as in item 3 of proposition 4.2: let $y_0, y_1, a_1, a_2, \ldots, a_n$ be as in the proposition. We define a second near-identity nilpotent pseudogroup $\tilde{f}_1, \ldots, \tilde{f}_N$ of functions satisfying $\tilde{f}_i(y_0) = y_0$, $\tilde{f}_i(y_1) = y_1$, $N = \binom{n}{2} + n - 1$.

Assume without loss of generality that $a_n > 0$. The construction of $\tilde{f}_i$, $i < n$, is similar to that of $F_j^{x_0}$ in section 4: define $F_{i,0} = \mathrm{id}$ and

$$F_{i,j+1} = f_n^{-\lfloor k/a_n \rfloor} \circ f_i \circ F_{i,j} \quad \text{where} \quad f_i(F_{i,j}(y_0)) = y_k.$$

In this way $F_{i,j}(y_0) \geq y_0$ for all i, j. We define $\tilde{f}_i = F_{i,a_n}$: we have $\tilde{f}_i(y_0) = y_0$ and $\tilde{f}_i(y_1) = y_1$. The remaining $\tilde{f}_i$ are the commutators $[f_j, f_k]$, $1 \leq j < k \leq n$. We claim that the functions $\tilde{f}_i$ commute: this will establish our claim that the original pseudogroup is metabelian.

Apply proposition 4.2 to the pseudogroup $\tilde{f}_i$ at some point in the interval (y_0, y_1): if we have cases 1 or 2 we are done. We assume therefore that we have case 3 in a maximal interval $(a, b) \subseteq (y_0, y_1)$. Let $\tilde{y}_0, \tilde{y}_1, \tilde{a}_1, \ldots, \tilde{a}_N$ be as in proposition 4.2. Assume without loss of generality that $\tilde{a}_k > 0$ so that $\tilde{f}_k(a) = a$, $\tilde{f}_k(b) = b$ and $\tilde{f}_k(x) > x$ for all $x \in (a, b)$. Assuming that the functions $\tilde{f}_i$ do not commute, let g be a commutator of highest order which is different from the identity. By construction $[\tilde{f}_k, g] = \mathrm{id}$. If g is not one of the $\tilde{f}_i$ then $g(\tilde{y}_0) = \tilde{y}_0$ so that g has a fixed point in (a, b). Lemma 5.1 now implies $g = \mathrm{id}$ which is a contradiction unless $g = \tilde{f}_i$ and the functions $\tilde{f}_i$ commute, as required.

6 Proof of theorems 1 and 2

We proved in the previous section that any near-identity pseudogroup is metabelian.

If at least one common fixed point exists, we are in case 1. Let $I_1 = (a, c)$ be a maximal open interval containing no common fixed points. Assume without loss of generality that $|a| < 1$ and apply proposition 2.1 to the interval (a, c) to obtain $b \in (a, c)$, a fixed point for all commutators in the pseudogroup. Let $f = f_j$ or $f = f_j^{-1}$ be such that $f(b)$ is maximal; from the results of section 3, $f(x) > x$ for all $x \in (a, c)$. Take g to be an arbitrary commutator and apply Koppel's lemma (5.1) to obtain $g(x) = x$ for all $x \in (a, c)$. This implies that the pseudogroup is abelian. For each i, let $a_i = \tau(f_i, f, b)$. If at least one of the a_i is irrational, the existence of the homeomorphism ϕ follows from Denjoy's theorem. Otherwise, we may write $a_i = p_i/q$, where p_i and q are integers. Assume without loss of generality that $p_k = 1$ so that, for each i, $\tau(f_i f_k^{-p_i}, f, b) = 0$: this implies the existence of $\tilde{b}_i \in [b, f(b)]$ with $f_i f_k^{-p_i}(\tilde{b}_i) = \tilde{b}_i$. Koppel's lemma now implies that $f_i = f_k^{p_i}$ and the pseudogroup consists of integer powers of f_k, implying the existence of the homeomorphism ϕ. This concludes the discussion of case 1.

If no common fixed point exists, take $x_0 \in X$ and let $f = f_j$ or $f = f_j^{-1}$ be such that $f(x_0)$ is maximal and set $a_i = \tau(f_i, f, x_0)$. If at least one of the a_i is irrational then from section 4 the pseudogroup is abelian and from Denjoy's theorem there exists a homeomorphism ϕ as stated, concluding case 2. Otherwise we are in case 3 of proposition 4.2, concluding theorem 1.

For theorem 2, take the holonomy pseudogroup and apply theorem 1: each case in one theorem corresponds to the same case in the other. In cases 1 and 2, the homomorphism $\alpha : G \to \mathbb{R}$ takes each generator γ_i of $\pi_1(G/H)$ to a_i as constructed in theorem 1 where $f_i = f_{\gamma_i}$. The homeomorphism Φ is constructed from ϕ. More precisely, for $(gH, x) \in G/H$ let $\gamma : [0, 1] \to G$ be a path with $\gamma(0) \in H$ and $\gamma(1) = g$. Lift γ to $\mathcal{F}_\alpha$ to obtain a path $\tilde{\gamma} : [0, 1] \to G/H \times \mathbb{R}$ tangent to $\mathcal{F}_\alpha$ with $\tilde{\gamma}(1) = (gH, x)$: let $\tilde{\gamma}(0) = (H, x_0)$. Now lift γ to $\mathcal{F}_1$ to obtain $\hat{\gamma} : [0, 1] \to G/H \times (-1, 1)$ with $\hat{\gamma}(0) = (H, \phi(x_0))$ and define $\Phi(gH, x) = \hat{\gamma}(1)$. The properties of ϕ imply that $\Phi(gH, x)$ is well defined, i.e., does not depend on the choice of γ. This concludes the proof of theorem 2.

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