Entangled States

George Svetlichny

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Consider a physical system consisting of two parts, each one of which is described by vectors in their respective Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The compound system is described by vectors of the tensor product of the two spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. A vector of the form $|\Psi\rangle = |\alpha \otimes \beta\rangle = |\alpha\rangle|\beta\rangle$ is called a *product vector*. An arbitrary vector of $\mathcal{H}$ is a linear combination (maybe infinite) of product vectors

$$|\Psi\rangle \;=\; \sum_i c_i |\alpha_i\rangle|\beta_i\rangle. \tag{1}$$

If we introduce the orthonormal basis $|\phi_i\rangle$ for $\mathcal{H}_1$ and $|\psi_j\rangle$ for $\mathcal{H}_2$, then the doubly indexed family $|\phi_i\rangle|\psi_j\rangle$ will be an orthonormal basis for $\mathcal{H}$ and hence

$$|\Psi\rangle = \sum_{ij} c_{ij} |\phi_i\rangle|\psi_j\rangle \tag{2}$$

for some coefficients c_{ij}. If now $|\eta_i\rangle = \sum_j c_{ij}|\beta_j\rangle$ then we have again the simple sum $|\Psi\rangle = \sum_i |\phi_i\rangle|\eta_i\rangle$ where the $|\phi_i\rangle$ are orthogonal, but the $|\eta_i\rangle$ in general are not. We shall see however that for and adequate choice of the $|\phi_i\rangle$, the $|\eta_i\rangle$ shall also be orthogonal.

For a product state $|\alpha\rangle|\beta\rangle$ we can say that the first part is in the pure state $|\alpha\rangle$ and the second in the pure state $|\beta\rangle$. In other words each part is in a state of its own independent of the other part. A state of the compound system that is not a product state is called an *entangled state*. For such a state, one cannot attribute an independent pure state for each of the parts. It's common to say that the parts do not posses physical states of their own.

An operator of the form $A \otimes I$ or $I \otimes B$ where A is a operator in $\mathcal{H}_1$ and B a operator in $\mathcal{H}_2$ is called a *local operator*. Observables of this form correspond to measurements made only upon one of the two parts of the compound system. A self-adjoint operator is called *maximal* if its spectrum is not degenerate. A local operator $A \otimes I$ never is maximal unless $\mathcal{H}_2$ is one dimensional. Such a local operator is called *locally maximal* if A is maximal.

A state $|\Psi\rangle \in \mathcal{H}$ possesses a local property if for some local observable $A \otimes I$ we have $(A \otimes I)|\Psi\rangle = \lambda|\Psi\rangle$.

Theorem 1 *If $|\Psi\rangle$ has a local property corresponding to the observable $A \otimes I$ then, following the notation of the previous paragraph, $|\Psi\rangle \in (P_\lambda \mathcal{H}_1) \otimes \mathcal{H}_2$, where P_λ is the orthogonal projector upon the subspace of eigenvectors of eigenvalue λ.*

Proof: In (2) let $|\phi_i\rangle$ be a basis composed of eigenvectors of A. Equating the two sides of $(A \otimes I)|\Psi\rangle = \lambda|\Psi\rangle$ we see that only the terms where $A|\phi_i\rangle = \lambda|\phi_i\rangle$ survive ■

Corollary 1 *A state $|\Psi\rangle$ has a local property corresponding to a local maximal observable if and only if is product state* ■

Starting from $|\Psi\rangle$ we define the two *partial traces* $\rho^{(1)} = \mathrm{Tr}^{(2)}(|\Psi\rangle\langle\Psi|)$ and $\rho^{(2)} = \mathrm{Tr}^{(1)}(|\Psi\rangle\langle\Psi|)$ of the operator $|\Psi\rangle\langle\Psi|$ by the requirement that for any operator A in $\mathcal{H}_1$ and B in $\mathcal{H}_2$ one has

$$\langle\Psi|A \otimes I|\Psi\rangle = \mathrm{Tr}(\rho^{(1)} A) \qquad (3)$$

$$\langle\Psi|I \otimes B|\Psi\rangle = \mathrm{Tr}(\rho^{(2)} B) \qquad (4)$$

It is easy to verify that if $|\Psi\rangle$ is normalized then the two partial traces are density matrices. From here on, $|\Psi\rangle$ will always be normalized.

An interpretation of these density matrices is immediate: restricting our measurements of the compound system to only one of the parts, the pure state represented by $|\Psi\rangle$ behaves in relation the these types of measurement exactly as a mixed state represented by the corresponding partial trace. One must not conclude however that each part *is* in a mixed state in any ontological sense.

It is easy to verify that in relation to expression (1) for $|\Psi\rangle$ we have:

$$\rho^{(1)} = \sum_{ij} c_i \bar{c}_j \langle\beta_j|\beta_i\rangle |\alpha_i\rangle\langle\alpha_j| \qquad (5)$$

$$\rho^{(2)} = \sum_{ij} c_i \bar{c}_j \langle\alpha_j|\alpha_i\rangle |\beta_i\rangle\langle\beta_j| \qquad (6)$$

and also in relation to (2) we have:

$$\rho^{(1)} = \sum_{ik}\sum_{m} c_{im}\bar{c}_{km}|\phi_i\rangle\langle\phi_k| \tag{7}$$

$$\rho^{(2)} = \sum_{j\ell}\sum_{n} c_{nj}\bar{c}_{n\ell}|\psi_j\rangle\langle\psi_\ell| \tag{8}$$

$$\tag{9}$$

Using $|\Psi\rangle$ we can define the *anti-linear* map $\Lambda : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ by requiring that $\langle\Lambda\alpha|\beta\rangle = \langle\Psi|\alpha\otimes\beta\rangle$. Its adjoint $\Lambda^\dagger$ is defined by $\langle\Lambda\alpha|\beta\rangle = \overline{\langle\alpha|\Lambda^\dagger\beta\rangle}$

Theorem 2

$$\rho^{(1)} = \Lambda^\dagger\Lambda \tag{10}$$

$$\rho^{(2)} = \Lambda\Lambda^\dagger \tag{11}$$

Proof: We prove (10), the proof of (11) is analogous. We must show that for all operator A in $\mathcal{H}_1$ one has $\mathrm{Tr}(\Lambda^\dagger\Lambda A) = \langle\Psi|A\otimes I|\Psi\rangle$. It's enough to show this for operators of the type $A = |a\rangle\langle b|$. Let now $|\gamma_j\rangle$ be an orthonormal basis for $\mathcal{H}_2$. We have

$$A\otimes I = \sum_j |a\otimes\gamma_j\rangle\langle b\otimes\gamma_j|$$

from which we deduce that

$$\langle\Psi|A\otimes I|\Psi\rangle = \sum_j \langle\Psi|a\otimes\gamma_j\rangle\langle b\otimes\gamma_j|\Psi\rangle =$$
$$\sum_j \langle\Lambda a|\gamma_j\rangle\overline{\langle\Lambda b|\gamma_j\rangle} = \langle\Lambda a|\Lambda b\rangle = \langle b|\Lambda^\dagger\Lambda|a\rangle = \mathrm{Tr}(\Lambda^\dagger\Lambda A).$$

A proof relative to $\rho^{(2)}$ is analogous $\blacksquare$

From the previous theorem we deduce also the following two useful relations:

$$\Lambda\rho^{(1)} = \rho^{(2)}\Lambda \tag{12}$$

$$\Lambda^\dagger\rho^{(2)} = \rho^{(1)}\Lambda^\dagger \tag{13}$$

Let now $\rho^{(1)}|\phi\rangle = r|\phi\rangle$. We have $\langle\Lambda\phi|\Lambda\phi\rangle = \overline{\langle\phi|\Lambda^\dagger\Lambda|\phi\rangle} = \overline{\langle\phi|\rho^{(1)}|\phi\rangle} = r\langle\phi|\phi\rangle$ seing that r is real. Also $\rho^{(2)}\Lambda|\phi\rangle = \Lambda\rho^{(1)}|\phi\rangle = r\Lambda|\phi\rangle$. Thus we've proved:

Theorem 3 *The density matrices $\rho^{(1)}$ and $\rho^{(2)}$ have the same non-null spectrum and for a non-null eigenvalue, Λ and $\Lambda^\dagger$ transforms eigenvector to eigenvector* $\blacksquare$

Theorem 4 *The space $\Lambda^\dagger\mathcal{H}_2$ is invariant for $\rho^{(1)}$ and we have $\mathcal{H}_1 = \overline{\Lambda^\dagger\mathcal{H}_2} \oplus \mathcal{N}(\rho^{(1)})$*

Proof: Let $|\beta\rangle \in \mathcal{H}_2$. The invariance of the subspace follows from $\rho^{(1)}\Lambda^\dagger|\beta\rangle = \Lambda^\dagger\rho^{(2)}|\beta\rangle$. Let now $|\alpha\rangle \in \mathcal{N}(\rho^{(1)})$ and $|\beta\rangle \in \mathcal{H}_2$. We have $\rho^{(1)}|\alpha\rangle = \Lambda^\dagger\Lambda|\alpha\rangle = 0$, hence $\langle\alpha|\Lambda^\dagger\Lambda|\alpha\rangle = \overline{\langle\Lambda\alpha|\Lambda\alpha\rangle} = 0$ and therefore $\Lambda|\alpha\rangle = 0$. Now $\langle\beta|\Lambda|\alpha\rangle = \overline{\langle\Lambda^\dagger\beta|\alpha\rangle} = 0$ which implies $\alpha \perp \Lambda^\dagger\mathcal{H}_2$ and thus $\alpha \perp \overline{\Lambda^\dagger\mathcal{H}_2}$. Reciprocally, from $\alpha \perp \overline{\Lambda^\dagger\mathcal{H}_2}$ we deduce that for all $|\beta\rangle \in \mathcal{H}_2$ one has $0 = \langle\Lambda^\dagger\beta|\alpha\rangle = \overline{\langle\beta|\Lambda\alpha\rangle}$. Therefore $\Lambda|\alpha\rangle = 0$ and finally $\rho^{(1)}|\alpha\rangle = \Lambda^\dagger\Lambda|\alpha\rangle = 0$ $\blacksquare$

Theorem 5 $|\Psi\rangle \in \overline{\Lambda^\dagger\mathcal{H}_2} \otimes \overline{\Lambda\mathcal{H}_1}$

Proof: Let $|\alpha\rangle \perp \Lambda^\dagger\mathcal{H}_2$. We have for any $|\beta\rangle \in \mathcal{H}_2$ that $\langle\Psi|\alpha \otimes \beta\rangle = \overline{\langle\alpha|\Lambda^\dagger\beta\rangle} = 0$. The same argument with $|\alpha\rangle$ arbitrary and $|\beta\rangle \perp \Lambda\mathcal{H}_1$ leads now the desired conclusion $\blacksquare$

Choose now a basis for the space $\mathcal{H}_1$ made of eigenvectors of $\rho^{(1)}$. By Theorem 4 we see that the elements of this basis that correspond to non-null eigenvalues generate the subspace $\overline{\Lambda^\dagger\mathcal{H}_2}$. Let $|\alpha_i\rangle$ be an enumeration of such elements and $r_i \neq 0$ the corresponding eigenvalues. We have $\langle\Lambda\alpha_i|\Lambda\alpha_j\rangle = \overline{\langle\alpha_i|\Lambda^\dagger\Lambda|\alpha_j\rangle} = r_i\delta_{ij}$ seeing that $\Lambda^\dagger\Lambda = \rho^{(1)}$. By Theorem 3 we conclude that $|\beta_i\rangle = (r_i)^{-1/2}|\Lambda\alpha_i\rangle$ is an orthonormal basis for the space $\overline{\Lambda\mathcal{H}_1}$. Now $\langle\Psi|\alpha_i \otimes \beta_j\rangle = \langle\Lambda\alpha_i|\beta_j\rangle = (r_i)^{1/2}\delta_{ij}$ and taking into account Theorem 5, we deduce the *bi-orthogonal expansion*:

$$|\Psi\rangle = \sum_i \sqrt{r_i}|\alpha_i\rangle|\beta_i\rangle. \tag{14}$$

A bi-orthogonal expansion of $|\Psi\rangle$ differs from that given by (2) in that the sum in simple instead of double.

Theorem 6 *A bi-orthogonal expansion is unique (up to permutations and multiplication of the $|\alpha_i\rangle$ and $|\beta_i\rangle$ by conjugate phases) if and only if the numbers r_i are all different, that is, if and only if $\rho^{(1)}$ (and therefore $\rho^{(2)}$) has a non-degenerate non-null spectrum.*

Proof: By construction, if the non-null spectrum were degenerate, the $|\alpha_i\rangle$ would not be unique and therefore the bi-orthogonal expansion would not be unique. On the other hand, from the bi-orthogonal expansion (14), we calculate that $\rho^{(1)} = \sum_i r_i |\alpha_i\rangle\langle\alpha_i|$. Therefore if the expansion were not unique, $\rho^{(1)}$ would be diagonalizable (in the subspace orthogonal to its nucleus) in different orthonormal bases (different set of projectors $|\alpha_i\rangle\langle\alpha_i|$) which implies that the non-null spectrum would be degenerate ■

Let us remind ourselves now of the Einstein-Podolsky-Rosen argument. They considered a state such that there is an exact correlation between the measured results corresponding to a pair of local operators $A \otimes I$ and $I \otimes B$ and at the same time an exact correlation between the measured results corresponding to another pair $A' \otimes I$ and $I \otimes B'$ being that A and A' do not commute. Suppose now that the four operators are maximal and $|\Psi\rangle$ is an entangled state. Suppose also (without loss of generality) that $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same dimension and that $\rho^{(1)}$ has a trivial nucleus (and therefore the same for $\rho^{(2)}$).

Theorem 7 *The correlations described in the previous paragraph are possible if and only if $\rho^{(1)}$ has a degenerate spectrum.*

Proof: Suppose that the observation corresponding to $A \otimes I$ is made first followed immediately later by $I \otimes B$. After the first measurement the state becomes projected upon a product state $|\alpha\rangle|\beta\rangle$ where $|\alpha\rangle$ is an eigenvector of A. For there to follow a certain result of measurement corresponding to $I \otimes B$, $|\beta\rangle$ must necessarily be an eigenvector of B. By the spectral theorem $|\Psi\rangle$ is the sum $\sum s_i |\alpha_i\rangle|\beta_i\rangle$ of product vectors obtained in this manner, that is, we have a bi-orthogonal expansion. Thus the same type of correlation could exist for another pair of observables incompatible with the first pair, if and only if the bi-orthogonal expansion were not unique. By Theorem 6 this happens if and only if $\rho^{(1)}$ has a degenerate spectrum ■