Subsumptions in the Formalism of Statistical Empirical Sciences

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Abstract

We show how the structures of formalized statistical sciences as presented in [1] can be subsumed in more primitive structures [2, 3] by abolishing some of the original distinctions. One of the byproduct of this process is a formalization of figure-ground relationships reminiscent of gestalt notions. Knowledge of [1] is assumed.

1 Introduction

In Reference [1] we developed a general formalism for statistical sciences, axiomatizing the processes of state preparation and state copy testing. We've paid for our attention to the details of empirical procedures by the complexity of the formalism. More elegant formalisms exist at the cost of ignoring some distinctions. Among the various formalisms for empirical sciences that exist in the literature, the one closest to ours is most probably that of empirical logic of the Amherst school [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] . Empirical logic is based on the primitive notion of an operation; that is, a series of acts which invariably lead to one and only one outcome from a previously defined set. No formal distinction is made as to whether such a procedure is one which prepares a state or one which performs a test on a state already prepared. Also, no formal distinction is made between what we in Reference [1] call an observation and what we call an exit. In this paper we show that a general formal statistical

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theory can be subsumed within a structure where these distinctions are abolished. We first abolish the distinction between an observation and an exit by introducing a formal background state to which a copy of a state is transferred in case an observation is realized in a test. As a second step we eliminate state preparations by considering them as certain types of exits.

There is a surprising richness in the first subsumption, at the heart of which we detect a formalization of figure-ground relationships reminiscent of some ideas of gestalt psychology. This is a welcome byproduct of the rather technical construction that we elaborate in the first section. The second subsumption, treated in the second section is almost trivial to implement.

One sees in present day literature a remarkably large number of formalizations of empirical scientific methodology. The complexity of the theme can only partially explain this fact. Another phenomenon seems also to be involved. Each formalism expresses, so to speak, only a certain aspect of the methodology and downplays others. The various formalisms seem to be largely connected by relations of mutual subsumption. One notices that one formalism is often definable in another and vice versa. This state of affairs, confusing at the beginning, is rather fortunate, since the details of effecting such subsumptions clarify, by explicit mathematical structures, a certain set of otherwise only linguistically formulated ideas whose direct formulation would leave something to be desired. The appearance of gestalt concepts in our first subsumption is a case in point. Apparently certain constructions need to reside in a more comprehensive framework, and only by existing as a sort of mathematical epiphenomena give any insight into what they model. It is in this spirit that we present our subsumptions.

Knowledge of Reference [1] is assumed. Reference thereto shall be designated by prefixing “I” to various items; thus “Theorem I.3” refers to Theorem 3 of that work. Certain of our results have straightforward but tedious demonstrations, and such are not presented in this paper. The author will gladly supply any interested reader with these details.

2 First Subsumption: operationalization

Consider a formal statistical theory (Definition I.3). We wish to abolish the distinction between an observation and an exit by introducing a new fictitious state ω which we whimsically refer to as either the *useless* or the *background* state, and consider that a state copy gets transformed to ω after observations, whereas exits maintain their previous properties. It seems plausible that such a construction can be carried out, and has been done so in other situations [14]. However, before doing so, we must first modify the formalism of Reference [1] to allow only exits. The first modification is in the concept of form algebras (Definition I.1 and I.2).

Definition 1 *An* operation form algebra G *is a universal algebra with three binary operations* $\vee$ *,* $+$*, and concatenation; and a distinguished subset* G_r*. The following axioms are satisfied:*

1. G_r *is a real associative algebra under (+,concatenation) with a unit which we denote by* $\mathbf{I}$*. We denote by* $\mathbf{0}$ *the zero of* G_r*. The domain of* $+$ *is* $G_r \times G_r$*.*
2. *The join* $\vee$ *is universally defined, free, associative, and commutative. By free we mean that every element of* G *can be written uniquely as* $x_1 \vee \cdots \vee x_n$ *with* $x_i \in G_r$*.*
3. *Concatenation is universally defined, has* $\mathbf{I}$ *for identity, is of course associative, and distributes over* $\vee$ *and* $+$ *on both sides.*

We write $G_{(n)}$ for the set of elements of type $x_1 \vee \cdots \vee x_n$.

These axioms are a straightforward adaptation, with some convenient algebraic simplification, of the axioms of Definition I.1.

One rather simple result is that given a form algebra F the purely operational part is an operation form algebra, that is, take $G_r = F_e$ and consider joins only of elements of F_e. One has to extend the domain of concatenation, but this offers no problems.

Definition 2 *The operation form algebra defined in the previous paragraph starting from a form algebra* F *we call, with a slight abuse of language, the* operation subalgebra *of* F*.*

Definition 3 *A* substitution operation form algebra *is a pair* (G, W) *where* G *is an operation form algebra and* W *is a real vector space. The following axioms are satisfied:*

1. *There is a bilinear map* $\langle \cdot, \cdot \rangle : W \times G \to \mathbb{R}$ *separating points of* W *and which sums on joins:* $\langle w, x_1 \vee \cdots \vee x_n \rangle = \sum_i \langle w, x_i \rangle$*.*
2. *There is an injective linear map from* W *to the space of right acting linear transformations* $G_r \to G_r$ *which commute with concatenation on the left by elements of* G_r*. We denote by* $\langle w, \cdot \rangle$ *the transformation corresponding to* $w \in W$*.*

3. $\langle u, x\langle w, \cdot\rangle\rangle = \langle u, x\rangle\langle w, \mathbf{I}\rangle$.

4. Given $y \in G_r$*, the adjoint map to concatenation by* y *defined by* $\langle w, yx\rangle = \langle y'w, x\rangle$ *satisfies* $y'W \subset W$ *and we have for* $x, y \in G_r$*,* $w \in W$*:*

$$x\langle w, \cdot\rangle y = x\langle y'w, \cdot\rangle.$$

This again is a straightforward adaptation of Definition I.2 where $\langle w, x\rangle$ stands for what in the old terminology would be written as $\langle w, x\mathbf{1}\rangle$.

In Reference [1], the separating condition of (1) was overlooked and should have been stated for technical convenience. A consequence of this condition is:

Proposition 1 *The map* $j : W \to G_r$ *given by* $jw = \mathbf{I}\langle w, \cdot\rangle$ *is injective.*

We note that since $x\langle w, \cdot\rangle = (x\mathbf{I})\langle w, \cdot\rangle = x(\mathbf{I}\langle w, \cdot\rangle)$ we need know only $\mathbf{I}\langle w, \cdot\rangle$ to define the action of $\langle w, \cdot\rangle$ on any element.

Given a substitution form algebra (F, V) one can consider the operational part of it as a substitution operation form algebra. The algebra G is as described at the end of Definition 1. Set $W = V$ and define the new pairing $\langle v, \cdot\rangle$ as $\langle v, \cdot\mathbf{1}\rangle$ extending $\langle\cdot, \cdot\rangle$ naturally over joins. The axioms are readily verified.

Definition 4 *The substitution operation form algebra defined in the previous paragraph we call the* operation subalgebra *of* (F, V)*.*

A statistical theory based only on operations can now be defined. An examination of Axioms I.1 to I.9 shows however that some modifications are necessary. The exhaustion axiom loses meaning. If an operation is not exhaustive, then it rejects some copies, but to be consistent in our outlook we must view the rejected copy as a transformed copy within our system, for otherwise rejection would be viewed as a type of observation and not as an exit. On the other hand, given a part of an operation there is no *a priori* way of completing it uniquely to an exhaustive one. There are thus two possible axiomatizations; either we deal only with exhaustive operations, forcing us to reduce the truncation axiom to the simple case of truncating dummies, or we maintain general operations and postulate the existence of *some* exhaustion of any given operation. Either one has certain small advantages but on the whole it doesn't seem to matter which one we choose. We shall adopt the first viewpoint to conform to what seems to be the prevailing attitude in the literature. In view of the remark following Proposition 1 we also find it convenient to recast the conversion Axiom I.6 into an existence axiom of state preparation.

Definition 5 *A* formal statistical operation theory *is a substitution operation form algebra* (G, W) *with the following additional structure:*

1. *A convex subset* $S \subset W$ *called the set of* states*;*
2. *A subset* $\hat{Q} \subset G$ *called the set of* operation forms.

We set $\hat{Q}_{(m)} = \hat{Q} \cap G_{(m)}$. *Given* $E = x_1 \vee \cdots \vee x_m \in \hat{Q}_{(m)}$ *we say each* x_i *is an* atom *or a* result *or an* outcome *of* E *and we write* $x_i \in E$. *Each subjoin* F *of* E *we call an* event *of* E *and we write* $F \subset E$. *We denote by* Q *the set of all events of operation forms and set* $Q_{(m)} = Q \cap G_{(m)}$.

These objects obey a series of axioms.

EXISTENCE AXIOMS:

Axiom 1 Stochastic splitters exist.
If $\Lambda = (\lambda_1, \ldots, \lambda_m)$, $0 \leq \lambda_i$, $\sum_i \lambda_i = 1$ *then* $Z_\Lambda = \lambda_1 \mathbf{I} \vee \cdots \vee \lambda_m \mathbf{I} \in \hat{Q}_{(m)}$.

Axiom 2 States can be prepared.
If $\sigma \in S$ *then* $\mathbf{I}\langle \sigma, \cdot \rangle \in \hat{Q}_{(1)}$.

TRANSFORMATION AXIOMS:

Axiom 3 Operation forms can be condensed.
Let $E \in \hat{Q}$, $F \subset E$, *and let* H *be the sum of the atoms of* F *then* $\mathbf{0} \vee (E \setminus F) \vee H \in \hat{Q}$.

Axiom 4 Dummies can be truncated.
Let $x \vee E \in \hat{Q}$ *and suppose* $x'S = \{\mathbf{0}\}$ *then* $E \in \hat{Q}$.

Axiom 5 Operation forms can be composed.
If $E, F \in \hat{Q}$ *and* x *is an atom of* E *then* $E\{x\}F = (E \setminus x) \vee xF \in \hat{Q}$.

STATISTICAL AXIOMS:

Axiom 6 States are weights.
Let $E \in \hat{Q}$, $F \subset E$ *and* $\sigma \in S$ *then* $\langle \sigma, F \rangle \in [0, 1]$ *and* $\langle \sigma, E \rangle = 1$.

Axiom 7 Exits are state transformers.
Let x *be an atom of* $E \in \hat{Q}$, *and* $\sigma \in S$. *If* $\langle \sigma, x \rangle \neq 0$ *then* $\dfrac{1}{\langle \sigma, x \rangle} x'\sigma \in S$.

Axiom 8 States are separated.
If for all $x \in \hat{Q}_{(1)}$, $\langle\sigma, x\rangle = \langle\tau, x\rangle$ *then* $\sigma = \tau$.

As in Reference [1], these axioms could also have been stated as derivation rules and one can talk, after the obvious straightforward modifications, of the corresponding deductions in the deductive system and the apparatus deductive system (Definition I.5).

It should be remarked that the statistical axioms are stable under the deductive system; that is, if we have a set of states and a set of forms satisfying Axioms 6-8 then any new forms that may be introduced by Axioms 1-5 continue satisfying Axioms 6-8. This fact was implicitly used in Reference [1] though never proved there. We formally state this result here, whose proof is straightforward.

Theorem 1 *The statistical axioms are stable under the deductive theory.*

There are two ways of viewing a formal statistical theory $T = (F, S, P)$ as a formal statistical operation theory. The first, utilizing Theorems I.1 and I.2, uses the operation subalgebra (G, V) of (F, V) and for $\hat{Q}$ takes the exhaustive operation forms of P. The reader can easily verify the axioms.

Definition 6 *The formal statistical operation theory described in the previous paragraph we shall call the* operation subtheory *of* T.

The second way is to perform a subsumption as shall be now described.

Following the intuitive idea of the background or useless state, let $W = V \oplus \mathbb{R}$ where we set $\omega = (0, 1)$ to correspond to the background state and so define the augmented set of states $\overline{S}$ to be the convex span of $S \cup \{\omega\}$ considering $\overline{S}$ as imbedded in W in the natural way. What relation must ω have to the original structure? Clearly for $p \in O_{(1)}$ we shall require that $p * \sigma = \omega$ for all σ for which $\langle\sigma, p\rangle \neq 0$. Without presenting a detailed argument here, it turns out that the only consistent universal way to assign numbers to $\langle\omega, p\rangle$ is to set $\langle\omega, p\rangle = 0$, including $\langle\omega, \mathbf{1}\rangle = 0$. This means also that every exit, including **Id** must be opaque to ω. The existence observation **1** and the do-nothing operation **Id** must thus lose their usual meaning and be replaced by yet another new entity. We thus introduce an element **I** which will become the unit of the subset G_r of an operation form algebra and which handles existence and do-nothing notions. Finally forms $\mathbf{I}\langle w, \cdot\rangle$, $w \in W$ are yet a third type of new entity since its action must detect existence, which $\mathbf{1}\langle w, \cdot\rangle$ no longer does. No further new elements are in fact now needed and we define:

$$G_r = F_o \oplus F_e \oplus W \oplus \mathbb{R} = F_o \oplus F_e \oplus V \oplus \mathbb{R} \oplus \mathbb{R}. \tag{1}$$

A general element $x \in G_r$ we express as (p, ψ, a, s, r) with $p \in F_o$, $\psi \in F_e$, $a \in V$; $s, r \in \mathbb{R}$. In particular we have $\mathbf{I} = (0, \mathbf{0}, 0, 0, 1)$ and $\mathbf{I}\langle (a, s), \cdot \rangle = (0, \mathbf{0}, a, s, 0)$. We also set $\Omega = (0, \mathbf{0}, 0, 1, 0)$. To save on notation we shall, until the end of this section, restrict the lower case letter p to represent an elements of F_o, the lower case Greek letter ψ , to represent an elements of F_e, and the lower case letters a, b to represent elements of V. Thus we can conventionally write $x = p + \psi + a + s\Omega + r\mathbf{I}$, where s and r are reals.

Define now the multiplication rule for concatenation by Table 1.

	$\hat{p}$	$\hat{\psi}$	$\hat{a}$	Ω
p	$\mathbf{0}$	$\mathbf{0}$	$p\langle \hat{a}, \cdot \rangle$	p
ψ	$\psi\hat{p}$	$\psi\hat{\psi}$	$\psi\mathbf{1}\langle \hat{a}, \cdot \rangle$	$\psi\mathbf{1}$
a	$\langle a, p \rangle$	$\hat{\psi}' a$	$\langle a, \mathbf{1} \rangle \hat{a}$	$\langle a, \mathbf{1} \rangle \Omega$
Ω	$\mathbf{0}$	$\mathbf{0}$	$\hat{a}$	Ω

Table 1: Concatenation

Define the pairing $\langle \cdot, \cdot \rangle : W \times G \to \mathbb{R}$ by Table 2.

$\langle \cdot, \cdot \rangle$	p	ψ	b	Ω	$\mathbf{I}$
a	$\langle a, p \rangle$	$\langle a, \psi\mathbf{1} \rangle$	$\langle a, \mathbf{1} \rangle \langle b, \mathbf{1} \rangle$	$\langle a, \mathbf{1} \rangle$	$\langle a, \mathbf{1} \rangle$
Ω	0	0	$\langle b, \mathbf{1} \rangle$	1	1

Table 2: Pairing

The reader is invited to convince himself or herself that these tables constitute the unique possible assignments consistent with the viewpoint expounded above. It is now possible to calculate $x'(\cdot)$ which we give in Table 3.

$\cdot'(\cdot)$	b	Ω
p	$\langle b, p \rangle$	0
ψ	$\psi' b$	0
a	$\langle b, \mathbf{1} \rangle a$	a
Ω	$\langle b, \mathbf{1} \rangle \Omega$	Ω
$\mathbf{I}$	b	Ω

Table 3: Adjointness

Introduce now a free join $\vee$ and extend the structures already defined in G_r to an operation form algebra and the pairing $\langle\cdot,\cdot\rangle$ to $\langle\cdot,\cdot\rangle : W \times G \to \mathbb{R}$ in the natural fashion.

Theorem 2 *The pair (G, W) defined above is a substitution operation form algebra.*

Definition 7 *The substitution operation form algebra (G, W) constructed above we call the* operationalization *of the substitution form algebra (F, V).*

We now turn our attention to a statistical theory $T = (F, V, P)$ and construct the corresponding statistical operation theory $\overline{T}$ in (G, W). As was already discussed, we have $\overline{S} = \mathrm{conv}(S \cup \{\omega\})$. To each test form

$$\Theta = p_1 \vee \cdots \vee p_n \vee \theta_1 \vee \cdots \vee \theta_m \in P$$

we must associate an operation form in $\hat{Q}$. Now reinterpreting each atom of Θ as an element of G_r is not enough; first of all Θ may not be exhaustive, so we must first pass on to $\hat{\Theta} = (\mathbf{1} - p_1 + \ldots - p_n - \theta_1\mathbf{1} + \ldots - \theta_m\mathbf{1}) \vee \Theta$, but still, with $\hat{\Theta}$ exhaustive, the same expression in G is not exhaustive since each element annihilates ω . We set

$$\overline{\Theta} = (\mathbf{I} - \mathbf{Id}) \vee \hat{\Theta}.$$

The element $\mathbf{I} - \mathbf{Id} = (0, -\mathbf{Id}, 0, 0, 1)$ represents that part of the do-nothing notion which pertains to G and not to F. In particular, $\overline{\mathbf{Id}} = (\mathbf{I} - \mathbf{Id}) \vee \mathbf{Id}$ is a two-exit operation which separates out the figure from the background, or the useful from the useless. The set of elements of the form $\overline{\Theta}$, $\Theta \in P$ do not satisfy the axioms of a statistical operation theory since condensations of observations and exits were not allowed in T and are now allowed. To form $\overline{T}$ we must therefore generate new forms. Let therefore $\hat{Q}$ be the set of operation forms generated by the deductive system of operation theories starting with the forms $\overline{\Theta}$, $\Theta \in P$.

Theorem 3 *The system $(G, W, \overline{S}, Q)$ satisfies the axioms of a formal statistical operation theory.*

Definition 8 *We call the formal statistical operation theory $\overline{T}$ defined above the* operationalization *of the formal statistical theory T.*

Having subsumed the general theories in operation theories, we must now provide criteria by which an operation theory can be identified as a subsumed general theory, being thus able to descend to the previous level of specific detail. A surprising amount of complexity appears in such a descent. Our criteria could possibly be simplified, however our aim in not to achieve maximal elegance at the moment, but to provide a demonstration that a return is possible with criteria that exhibit somehow the essence of the subsumption process as detected in the resulting structure. We first treat substitution operation form algebras.

For the next five propositions we assume that (G, W) is the operationalization of a substitution form algebra (F, V).

Proposition 2 *Subalgebra G_r contains elements U, L, and D satisfying the multiplication law given by Table 4.*

	U	L	D
U	U	D	D
L	$\mathbf{0}$	L	$\mathbf{0}$
D	$\mathbf{0}$	D	$\mathbf{0}$

Table 4: Figure-ground or Utility Subalgebra

Proof. Put $U = \mathbf{Id}$, $L = \Omega$, $D = \mathbf{1}$, and apply Table 1. We list here some useful properties of these operators:

$$\begin{aligned}
Ux &= p + s\mathbf{1} + \psi + \mathbf{1}\langle a, \cdot\rangle + r\mathbf{Id} && (2)\\
xU &= \psi + r\mathbf{Id} + a && (3)\\
Lx &= a + (s + r)\Omega && (4)\\
xL &= p + \psi\mathbf{1} + (\langle a, \mathbf{1}\rangle + s + r)\Omega && (5)\\
Dx &= (s + r)\mathbf{1} + \mathbf{1}\langle a, \cdot\rangle && (6)\\
xD &= \psi\mathbf{1} + r\mathbf{1} + \langle a, \mathbf{1}\rangle && (7)\\
U'(a, s) &= (a, 0) && (8)\\
L'(a, s) &= (0, \langle a, \mathbf{1}\rangle + s) && (9)\\
D'(a, s) &= (0, \langle a, \mathbf{1}\rangle) && (10)
\end{aligned}$$

We see that U acts as a filter for figures, L is the preparation of the background or useless state, and D is a degradation operator, annihilating the ground, and declaring the figure as ground. Within empirical sciences, the above elements represent certain gross procedures; U corresponds to the isolation of a prepared state of affairs from surrounding elements before subsequent detailed investigations, L corresponds to "cleaning the slate", that is, returning to an undifferentiated background situation, and D corresponds to considering a given state of affairs as no longer relevant to the investigation. These elements also formalize certain figure-ground relationships familiar in gestalt psychology.

Definition 9 *An algebra with unit* $\mathbf{I}$ *and generated by elements satisfying the multiplication law of Table 4 we call a* figure-ground*, or* utility *algebra.*

Proposition 3 *The idempotent* $M = \mathbf{I} - U - L + D$ *is central of corank* 1.

Phenomenologically, M is rather mysterious; it somehow incorporates the conventions that we've adopted: that $\mathbf{I}$ takes over the role of $\mathbf{Id}$ and the role of $\mathbf{1}$ in the operationalization process.

Proposition 4 *For all* $w \in W$, $L\langle w, \cdot\rangle = \mathbf{I}\langle w, \cdot\rangle$.

Phenomenologically this equality means that if we detect existence, create the background state, detect the existence of this background state, and create a state w, then the same result obtains if we simply skip the intermediate creation of the background state.

By Table 4 we deduce from this proposition that for all $w \in W$ we have $D\langle w, \cdot\rangle = U\langle w, \cdot\rangle$.

Proposition 5 $LG_r(\mathbf{I} - U) = \mathbb{R}\{L\}$.

Phenomenologically this means that if we operate on the background state and then remove the figure, we get a multiple of the background state, that is, there is a *unique* background state.

Proposition 6 $F_o = UG_rL$, $F_e = UG_rU$, *and* $W = LG_rU \oplus \mathbb{R}\{L\}$.

Suppose now that (G, W) is a general substitution operation form algebra. We now consider a series of postulates imposed on (G, W) which in their totality will lead (G, W) to be isomorphic to an operationalization of a substitution form algebra.

Postulate 1 *G_r contains a utility subalgebra with generators U, L, and D satisfying the rules of Table 4.*

Postulate 2 *The idempotent $M = \mathbf{I} - U - L + D$ is central and has a kernel of codimension 1.*

Given $x \in G_r$, then since $\mathbf{I} \notin \ker(M)$, there is a unique real r such that $M(x - r\mathbf{I}) = \mathbf{0}$. Let $x_0 = x - r\mathbf{I}$, $y = x_0 - Lx_0U$, $z = y - UyU$, and $w = z - Lz(\mathbf{I} - U)$.

Proposition 7 *Under Postulates 1-2, $x = UwL + UyU + Lz(\mathbf{I} - U) + Lx_0U + r\mathbf{I}$*

Proof: We have in terms of x_0 and using utility algebra multiplication rules:

$$UwL = (U - D)x_0(L - D), \tag{11}$$

$$UyU = (U - D)x_0U, \tag{12}$$

$$Lz(\mathbf{I} - U) = Lx_0(\mathbf{I} - U). \tag{13}$$

Thus $UwL + UyU + Lz(\mathbf{I} - U) + Lx_0U = (U - D)x_0(L - D + U) + Lx_0 = (L - D + U)x_0 = x_0$, where we've used the fact that $Mx_0 = \mathbf{0} \Leftrightarrow (U - D + L)x_0 = x_0$. **Q.E.D.**

Proposition 8 *Under Postulates 1-2 the subspaces UG_rL, UG_rU, $LG_r(\mathbf{I} - U)$, LG_rU, and $\mathbb{R}\{\mathbf{I}\}$ are linearly independent.*

Proof: Assume $UxL + UyU + Lz(\mathbf{I} - U) + LwU + r\mathbf{I} = \mathbf{0}$. Multiplying by L on the left and by U on the right, we get $LwU = \mathbf{0}$ and so $UxL + UyU + Lz(\mathbf{I} - U) + r\mathbf{I} = \mathbf{0}$. Multiplying by U on the right, we get $UyU + rU = \mathbf{0}$ so we have $UxL + Lz(\mathbf{I} - U) + r(\mathbf{I} - U) = \mathbf{0}$. Multiplying by L on the left, we get $Lz(\mathbf{I} - U) + rL = \mathbf{0}$ and so $UxL + r(\mathbf{I} - U - L) = \mathbf{0}$ which multiplied by U on the left gives $UxL + r(-D) = \mathbf{0}$. Combining the last two equalities we conclude that $rM = \mathbf{0}$. By Postulate 2, $r = 0$, and this combined with previous equalities leads to linear independence. **Q.E.D.**

In view of Proposition 3, we now introduce:

Postulate 3 *For all $w \in W, L\langle w, \cdot\rangle = \mathbf{I}\langle w, \cdot\rangle$.*

Lemma 1 *Under Postulate 3, for all $w \in W$ and all $x \in G_r$, $\langle w, x\rangle = \langle w, xL\rangle$ and $\langle w, xU\rangle = \langle w, xD\rangle$.*

Proof: For all $u, v \in W$ we have: $\langle u, \mathbf{I}\rangle\langle v, \mathbf{I}\rangle = \langle u, \mathbf{I}\langle v, \cdot\rangle\rangle = \langle u, L\langle v, \cdot\rangle\rangle = \langle u, L\rangle\langle v, \mathbf{I}\rangle$, thus $\langle u, \mathbf{I}\rangle = \langle u, L\rangle$ since v is arbitrary. Now, $\langle w, x\rangle = \langle x'w, \mathbf{I}\rangle = \langle x'w, L\rangle = \langle w, xL\rangle$. Finally, $\langle w, xU\rangle = \langle w, xUL\rangle = \langle w, xD\rangle$. **Q.E.D.**

Let us now perform the decomposition given by Proposition 7 on the element $x = \mathbf{I}\langle w, \cdot\rangle$. By Postulate 3 we readily conclude that $Mx = \mathbf{0}$ so $x_0 = x$. Using Postulate 3 and the formulae in the proof of Proposition 4, we find: $UwL = \mathbf{0}$, $UyU = \mathbf{0}$, $Lx_0U = \mathbf{I}\langle U'w, \cdot\rangle$, and $Lx_0(\mathbf{I} - U) = \mathbf{I}\langle(\mathbf{I} - U)'w, \cdot\rangle$. Thus $\mathbf{I}\langle w, \cdot\rangle = \mathbf{I}\langle U'w, \cdot\rangle + \mathbf{I}\langle(\mathbf{I} - U)'w, \cdot\rangle$, the two parts lying in LG_rU, and $LG_r(\mathbf{I} - U)$ respectively. Let now $j : W \to G_r$ be the map $w \mapsto \mathbf{I}\langle w, \cdot\rangle$. We've thus shown that $jW \subset LG_rU \oplus LG_r(\mathbf{I} - U)$. Since in an operationalization we have an equality, we assume:

Postulate 4 *$jW = LG_rU \oplus LG_r(\mathbf{I} - U)$.*

In view of Proposition 5 we also adopt.

Postulate 5 *$LG_r(\mathbf{I} - U) = \mathbb{R}\{L\}$.*

Lemma 2 *Under Postulates 3–5, $x\langle w, \cdot\rangle yL = \langle w, y\rangle xL$.*

Proof: We have $x\langle w, \cdot\rangle yL = xL\langle w, \cdot\rangle yL$ by Postulate 4. Now, by utility algebra and Postulate 5, $L\langle w, \cdot\rangle L = L\langle w, \cdot\rangle L(\mathbf{I} - U) = sL$ for some real s. Let $v \in W$, then, using utility algebra, Lemma 1, and Postulate 3,

$$\langle v, L\langle w, \cdot\rangle yL(\mathbf{I} - U)\rangle = \langle v, \mathbf{I}\langle y'w, \cdot\rangle\rangle = \langle v, L\rangle\langle y'w, \mathbf{I}\rangle = \langle v, \mathbf{I}\rangle\langle w, y\rangle$$

Also $\langle v, sL\rangle = s\langle v, L\rangle = s\langle v, \mathbf{I}\rangle$ and since v is arbitrary, $s = \langle w, y\rangle$. **Q.E.D.**

Now if $jw_1 = jw_2$ then for all x, $\langle w_1, x\rangle = \langle w_2, x\rangle$ since for arbitrary v, $\langle v, \mathbf{I}\langle w_i, \cdot\rangle x\rangle = \langle v, \mathbf{I}\rangle\langle w_i, x\rangle$. Thus it makes sense to talk of the pairing $\langle\cdot, \cdot\rangle$ as also being defined on $jW \times G_r$. Now $L \in jW$ by Postulate 4 since $L = L\mathbf{I}(\mathbf{I} - U)$. In view of Table 2 we now assume:

Postulate 6 $\langle L, \mathbf{I}\rangle = 1$

Theorem 4 *Let (G, W) be a substitution operation form algebra satisfying Postulates 1–6, then it is isomorphic to the operationalization of a substitution form algebra (F, V).*

Proof: In view of Proposition 6 we define $F_o = UG_rL$ and $F_e = UG_rU$. By Postulate 4 and Proposition 7 we have $G_r = F_o \oplus F_e \oplus jW \oplus \mathbb{R}\{\mathbf{I}\}$. We also have by Postulate 5 that $jW = LG_rU \oplus \mathbb{R}\{L\}$. We can thus decompose W as $V \oplus \mathbb{R}$ with $jV = LG_rU$ and $j(0,1) = L$. The triple of spaces F_o, F_e, and V can now be given the structure of a substitution form algebra with $D = U\mathbf{I}L \in F_o$ identified with $\mathbf{1}$ and $U = UUU \in F_e$ identified with $\mathbf{Id}$. We maintain the join of G and notice that since $F_eF_o = UG_rUUG_rL \subset UG_rL = F_o$, we can maintain the concatenation also. The form algebra axioms (Definition I.1 and I.2) are readily verified. To prove that (G, W) is the operationalization of (F, V) a check must be made of the validity of Tables 1-2 which is quite straightforward using utility algebra, Lemmas 1-2 and the Postulates. **Q.E.D.**

Theorem 5 *Let $T = (G, W, \hat{Q})$ be a formal statistical operation theory in which (G, W) satisfies Postulates 1-6, then T is the operationalization of a formal statistical theory T_0 if and only if $(\mathbf{I} - U) \vee U \in \hat{Q}_{(2)}$.*

Phenomenologically this means that the theory allows for the separation of the figure from the ground as one of its operational procedures.

Necessity being obvious, we prove only sufficiency.

Proof: If $\sigma \in S$ then it can be written as $\langle \cdot, \mathbf{I}-U\rangle(\mathbf{I}-U)*\sigma + \langle \cdot, U\rangle U*\sigma$. Now, by Lemma 2 and Postulates 3 and 5 we have $\mathbf{I}\langle\sigma, \cdot\rangle(\mathbf{I} - U) = \mathbf{I}\langle(\mathbf{I} - U)'\sigma, \cdot\rangle L = \langle\sigma, (\mathbf{I}-U)\rangle L$. Thus $(\mathbf{I}-U)*\sigma = L$. We see therefore that $S = \mathrm{conv}(U*S \cup \{L\})$. Let $S_0 = U*S$. By Theorem 4, (G, W) is the operationalization of a substitution form algebra (F, V). We note that $S_0 \subset V$. Suppose now that $E = x_1 \vee \cdots \vee x_m \in \hat{Q}_{(m)}$. Let $x_i = p_i + \psi_i + t_i\langle\cdot_i, \cdot\rangle + s_i\Omega + r_i\mathbf{I}$. Composing we have

$$(\mathbf{I} - U)Ux_1 \vee \cdots \vee Ux_m = (\mathbf{I} - U) \vee \bigvee_{i=1}^{m} (p_i + \psi_i + t_i\mathbf{1}\langle\sigma_i, \cdot\rangle + s_i\mathbf{1} + r_i\mathbf{Id}),$$

and composing each of the Ux_i atoms with $(\mathbf{I} - U) \vee U$ we have that

$$(I - U) \vee \bigvee_{i=1}^{m} ((p_i + s_i\mathbf{1}) \vee (\psi_i + t_i\mathbf{1}\langle\sigma_i, \cdot\rangle + r_i\mathbf{Id})) \in \hat{Q}.$$

After truncating dummies, this has the form $(\mathbf{I}-U) \vee q_1 \vee \cdots \vee q_k \vee \theta_1 \vee \cdots \vee \theta_n$ with $q_i \in F_e$ and $\theta_j \in F_e$. Let P be the set of forms $E_0 = q_1 \vee \cdots \vee q_k \vee \theta_1 \vee \cdots \vee \theta_n$ that can be obtained this way, along with their truncations. We claim that $T_0 = (F, V, P)$ is a formal statistical theory. Let us check the axioms. Now, $(\mathbf{I} - U) \vee UZ_\Lambda = (\mathbf{I} - U) \vee (\lambda_1\mathbf{Id} \vee \cdots \vee \lambda_m\mathbf{Id})$ thus $\lambda_1\mathbf{Id} \vee \cdots \vee \lambda_m\mathbf{Id} \in P$ and stochastic splitters exist. The condensation, composition and conversion axioms are satisfied by virtue of their being true in T. The truncation axiom is satisfied by construction. The original form E_0 is exhaustive by virtue of the exhaustivity of E on S and the fact that $(\mathbf{I}-U)'S_0=\{0\}$. The exhaustion of some of its truncates is therefore a condensation of E_0 possibly with $\mathbf{1}$ placed on some of the exits. Now $\mathbf{1}\in P$ since $\mathbf{1}=E_0$ for $E=\mathbf{I}\langle L,\cdot\rangle$ and so the exhaustion axiom is satisfied. Now for $\sigma\in S_0$, $\langle\sigma,p\rangle\in[0,1]$ since it is true in T, also

$$\langle\sigma,\mathbf{1}\rangle=\langle\sigma,D\rangle=\langle\sigma,UL\rangle=\langle U'\sigma,L\rangle=\langle U'\sigma,\mathbf{I}\rangle=\langle\sigma,\mathbf{I}\rangle=1$$

since $\sigma\in U'S$; furthermore, σ transformed by an exit θ is again in S_0 since $\theta=UyU$ for some y and so $\theta'\sigma=U'y'U'\sigma\in U'S$; finally, we have $\langle\sigma,x\rangle=\langle U'\sigma,xL\rangle=\langle\sigma,UxL\rangle$ so F_o separates states in S_0. Thus (F,V,P) is a formal statistical theory. If $\Theta\in P$ then Θ is of the form E_0 for some $E\in\hat{Q}$. Thus to prove that T is $\overline{T}_0$ we need only show that E can be recovered from E_0 by the deductive system. Now, $1=\langle L,E\rangle=\sum_i(t_i+s_i+r_i)$. We can thus by fragmenting (Definition I.15) the first atom of $(\mathbf{I}-U)\vee U$ get an operation form

$$(\bigvee_{i=1}^{p}t_i(\mathbf{I}-U))\vee(\bigvee_{i=1}^{p}s_i(\mathbf{I}-U))\vee(\bigvee_{i=1}^{p}r_i(\mathbf{I}-U))\vee U.$$

By composition, seeing that $\mathbf{I}\langle\sigma_i,\cdot\rangle\in\hat{Q}$, and $L=\mathbf{I}\langle\omega,\cdot\rangle\in\hat{Q}$ we get an operation form

$$(\bigvee_{i=1}^{p}t_i(\mathbf{I}-U)\langle\sigma_i,\cdot\rangle)\vee(\bigvee_{i=1}^{p}s_i(\mathbf{I}-U)L)\vee(\bigvee_{i=1}^{p}r_i(\mathbf{I}-U))\vee UE_0.$$

Now $UE_0=\mathbf{0}\vee\bigvee_{i=1}^{p}Ux_i$, $U\langle\sigma_i,\cdot\rangle=UL\langle\sigma_i,\cdot\rangle=\mathbf{1}\langle\sigma_i,\cdot\rangle$, $(\mathbf{I}-U)L=L-D=\Omega-\mathbf{1}$. Dropping the dummy $\mathbf{0}$, we now get the form

$$(\bigvee_{i=1}^{p}(t_i\mathbf{I}\langle\sigma_i,\cdot\rangle-t_i\mathbf{1}\langle\sigma_i,\cdot\rangle))\vee(\bigvee_{i=1}^{p}(s_i\Omega-s_i\mathbf{1}))\vee$$
$$(\bigvee_{i=1}^{p}(r_i\mathbf{I}-r_i\mathbf{Id}))\vee(\bigvee_{i=1}^{p}(p_i+s_i\mathbf{1}+\psi_i+t_i\mathbf{1}\langle\sigma_i,\cdot\rangle+r_i\mathbf{Id})).$$

An appropriate condensation gives E. **Q.E.D.**

Example 1: Let $T=(F,V,P)$ be a formal statistical theory whose associated statistical theory is a classical n-dimensional Boolean theory. Now P must contain an operation form $X=x_1\vee\cdots\vee x_n$ which is the classical fractioning operation, separating each state into the pure fractions composing it. That is, in the standard geometric representation of Examples I.1 and I.2, X has the form $(e_1\langle\sigma^1,\cdot\rangle,\ldots,e_n\langle\sigma^n,\cdot\rangle)$ where

$$e_i = (0,\ldots,0,1,0,\ldots,0)\in O, \quad (14)$$
$$\sigma^i = (0,\ldots,0,1,0,\ldots,0)\in S \quad (15)$$

with 1 in the i-th place. In the operationalization $\overline{T}$ of T one has the operation $\overline{X}=(\mathbf{I}-U)\vee X$ and so also $(\mathbf{I}-U)L\vee X$. Now $\langle\sigma^i,\mathbf{I}-U)=0$, $\langle\omega,\mathbf{I}-U\rangle=1$

and $L = \mathbf{I}\langle\omega,\cdot\rangle$ so X is the fractioning operation for $\overline{S} = \mathrm{conv}(S \cup \{\omega\})$ which is an n-dimensional simplex. Thus by Theorem I.2, T has associated to it an $n+1$-dimensional classical Boolean statistical theory. This implies that a finite dimensional classical Boolean theory can be constructed by repeated operationalizations starting with the trivial one-dimensional theory possessing only one state. Notice that the operation version of the standard classical theory is not an operationalization since then G_r is the algebra of $n \times n$ real matrices and has a trivial center and so cannot posses the idempotent M of corank 1. This means that the present view of Boolean theories can only be maintained by keeping operationally distinct statistically indistinguishable elements. Classical Boolean theories can thus be viewed as either based on purely formal distinctions, or alternatively as those in which one can nest sharp figure-ground relations to a depth given by the dimension of the state space.

3 Second subsumption: elimination of state preparations

Suppose that (G, W) is a substitution form algebra and (G, W, Q) a formal operation statistical theory. The state space S can be conventionally eliminated by identifying it with the set of elements of the form $\mathbf{I}\langle\sigma,\cdot\rangle$, that is conversion operations to elements of S. We begin thus identifying W with $jW \subset G_r$.

Proposition 9 *jW is a right ideal of G_r with the property that if $\theta, \phi \in jW$ then $\theta\phi = \langle\theta\rangle\phi$ where $\langle\cdot\rangle : jW \to \mathbb{R}$ is a linear form satisfying the separation condition:*

$$(\forall x, \langle\theta x\rangle = \langle\phi x\rangle) \Rightarrow \theta = \phi.$$

Proof: By the proof of Proposition 1, $\mathbf{I}\langle w,\cdot\rangle\mathbf{I}\langle v,\cdot\rangle = \langle w,\mathbf{I}\rangle\mathbf{I}\langle v,\cdot\rangle$ so $\langle\cdot\rangle = \langle\cdot,\mathbf{I}\rangle$. **Q.E.D.**

Definition 10 *A* statistical ideal *$W \subset G_r$ of an operation form algebra is a right ideal with the property that $a, b \in W \Rightarrow ab = \langle a\rangle b$ for some linear functional $\langle\cdot\rangle$ on W satisfying the separation property of the penultimate paragraph.*

Proposition 10 *If W is a statistical ideal of an operation form algebra G then the pair (G, W) becomes a substitution operation form algebra, if we define the functional $\langle\cdot,\cdot\rangle : W \times G \to \mathbb{R}$ on $W \times G_r$ by $\langle w, x\rangle = \langle wx\rangle$ and extended to sum on joins, and if for $w \in W$ we define the map $\langle w,\cdot\rangle : G_r \to G_r$ as right multiplication: $x\langle w,\cdot\rangle = xw$.*

We can thus now give an alternative definition of a formal operation statistical theory.

Definition 5 $'$ *A formal operation statistical theory is an operation form algebra G with the following additional structures:*

1. *A convex subset S of a statistical ideal W of G.*
2. *A subset $\hat{Q} \subset G$ called the set of operation forms.*

Adopting the same conventions as in Definition 5 the following axioms are now assumed:

Axiom 1 $'$ *Same as Axiom 1.*

Axiom 2 $'$ $S \subset \hat{Q}_{(1)}$.

Axiom 3 $'$ *Same as Axiom 3.*

Axiom 4 $'$ *Let $x \vee E \in \hat{Q}$ and suppose $Sx = \{0\}$ then $E \in \hat{Q}$.*

Axiom 5 $'$ *Same as Axiom 5.*

Axiom 6 $'$ *Same as Axiom 6.*

Axiom 7 $'$ *If x is an atom of $E \in Q$ and $\sigma \in S$ is such that $\langle \sigma x \rangle \neq 0$ then $x * \sigma = \dfrac{\sigma x}{\langle \sigma x \rangle} \in S$.*

There is no axiom corresponding to Axiom 8 since its content has been incorporated into the definition of a statistical ideal.

We have not yet achieved an elimination of state preparation notions since the above axioms refer to a set of states S. We can however introduce a structure that simply eliminates those portions of the previous definition that refer to states and then perform a subsumption.

Definition 11 *Let G be an operation form algebra and $(G, \hat{Q})$ a structure defined by eliminating (1) and Axioms $2'$, $4'$, $6'$, and $7'$ in Definition $5'$. We call such a structure a* formal general operation statistical theory.

Let us now see how the structure so defined subsumes that of the previous one.

Suppose $(G, \hat{Q})$ is a formal general operation statistical theory and suppose $W \subset G_r$ is a statistical ideal. We claim that there is a largest set $S_0 \subset W \cap \hat{Q}_{(1)}$ for which the statements of Axioms 6' and 7' are true. Let C be the convex set of $w \in W \cap \hat{Q}_{(1)}$ for which the statement of Axiom 6' is true. For $w \in C$ using then the composition axiom and the fact that $\langle wE\{x\}F\rangle = \langle wE\rangle = 1$ one concludes that if $\langle wx\rangle \neq 0$ then $x * w \in C$.

Define now

$$S_0 = \{w \in C | x * w \in \hat{Q}_{(1)} \text{ if } \langle wx\rangle \neq 0 \text{ and } x \in E \in \hat{Q}\}.$$

Now Axioms 2' and 7' are satisfied for S_0. Firstly, we have $w \in S_0 \Rightarrow w = \mathbf{I} * w \in \hat{Q}_{(1)}$, which takes care of Axiom 2'; and secondly, if x is an atom of E and y of F and $\langle x * wy\rangle \neq 0$ then $\langle wxy\rangle \neq 0$ and $y * x * w = (xy) * w$, but xy is and atom of $E\{x\}F$ and so by definition of S_0, $y * (x * w) \in \hat{Q}_{(1)}$ and Axiom 7' is satisfied. Thus the structure defined by $(G, \hat{Q}, W, S_0)$ now satisfies all axioms except possibly Axiom 4'; this however can simply be appended as a derivation rule and the structure completed with respect to this rule.

Definition 12 *The formal operation statistical theory constructed above is called the* canonical theory *associated to the ideal W.*

We may thus forgo the notion of state preparation and study only theories that posses operations but for which the notion of a statistical state is foreign. Operations can still be construed as a sort of transformation of the "state of affairs", but we no longer have the power to create any *a priori* determined state of affairs. Social, economic and political sciences can possibly be viewed as being of this type.

Example 2: Let $G_r = M(2, \mathbb{R})$, the matrix algebra of 2×2 real matrices. One can easily verify that the right ideals of G_r are all of the form AG_r where A is factorizable into a tensor product of two vectors:

$$A = \begin{pmatrix} ca & cb \\ da & db \end{pmatrix} = (c, d) \otimes (a, b).$$

For $A \neq 0$ these are statistical ideals with $\langle Am\rangle = (a, b)m \begin{pmatrix} c \\ d \end{pmatrix}$

For statistical ideals we see that two such matrices define the same ideal if and only if the two corresponding vectors (c, d) define the same point in the real projective line, that is, if and only if they are colinear; the ideal itself is then $\{(c,d)\otimes(a,b)\,|\,(a,b)\in\mathbb{R}^2\}$. Any statistical theory will have its state space contained in the line $ca+db=1$ and would thus either be reduced to a point or be a segment, in which case we would be dealing with a two-dimensional theory (Example I.3).

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